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Higher-Order Corrections to Non-Compact Calabi-Yau Manifolds in String Theory

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ABSTRACT

At the leading order, the low-energy effective field equations in string theory admit solutions of the form of products of Minkowski spacetime and a Ricci-flat Calabi-Yau space. The equations of motion receive corrections at higher orders in α', which imply that the Ricci-flat Calabi-Yau space is modified. In an appropriate choice of scheme, the Calabi-Yau space remains Kähler, but is no longer Ricci-flat. We discuss the nature of these corrections at order ${\alpha'}^3$, and consider the deformations of all the known cohomogeneity one non-compact Kähler metrics in six and eight dimensions. We do this by deriving the first-order equations associated with the modified Killing-spinor conditions, and we thereby obtain the modified supersymmetric solutions. We also give a detailed discussion of the boundary terms for the Euler complex in six and eight dimensions, and apply the results to all the cohomogeneity one examples.

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Contents

1 Introduction

Calabi-Yau manifolds have played a central rôle in string theory, by providing the compactifying spaces that permit four-dimensional effective actions to be extracted from ten-dimensional strings, via a Kaluza-Klein mechanism [1]. In this context, the requisite Calabi-Yau manifolds are six-dimensional, and they must be compact so that the Kaluza-Klein spectrum will be discrete, with a mass gap. The special holonomy, $SU(3)$, of the Calabi-Yau spaces is a crucial aspect of their structure, since it implies that there will be $N = 1$ supersymmetry in the four-dimensional spacetime.

More recently, within the framework of the AdS/CFT correspondence, Calabi-Yau manifolds and other spaces of special holonomy that are instead *non-compact* have found a natural rôle. They can provide gravity duals for superconformal field theories with less than the maximal supersymmetry on the boundaries of anti-de Sitter spacetimes that arise in the decoupling limits of D-branes or M-branes.

At leading order, the effective equations of motion in string theory imply that a configuration of the form $(\text{Minkowski})_d \times K_{10-d}$ will give a solution if the "internal" manifold K_{10-d} is Ricci-flat. The further requirement of unbroken supersymmetry implies that it should have special holonomy. Beyond the leading order, there are correction terms in the effective action that modify the equations of motion that the background must satisfy. In particular, there are corrections, starting at order ${\alpha'}^3$, which imply in general that the internal manifold will no longer be Ricci-flat. This is the case even in situations with supersymmetry, such as when K is a Kähler manifold.

In this paper we study the effects of the ${\alpha'}^3$ corrections in detail for several examples of six-dimensional and eight-dimensional Kähler manifolds. The metrics that we consider are all of cohomogeneity one, which means that the Einstein equation, together with the higher-order corrections, gives rise to a system of coupled ordinary differential equations for metric functions. At leading order the metrics are Ricci flat. Our examples in six dimensions include the resolved and deformed conifolds, and the $\mathbb{R}^2$ bundle over $\mathbb{CP}^2$ or $S^2 \times S^2$. In eight dimensions we consider $\mathbb{R}^2$ bundles over $S^2 \times S^2 \times S^2$, $S^2 \times \mathbb{CP}^2$ or $\mathbb{CP}^3$; $\mathbb{R}^4$ bundles over $S^2 \times S^2$ or $\mathbb{CP}^2$; and the Stenzel metric on the $\mathbb{R}^4$ bundle over S^4. (The ${\alpha'}^3$ corrections for the six dimensional resolved and deformed conifolds, and the $\mathbb{R}^2$ bundle over $S^2 \times S^2$, were previously studied in [2].) In each case we derive first-order systems of equations that describe the corrections to Ricci-flatness implied by the ${\alpha'}^3$ terms in the string effective action. We obtain a general implicit solution of the corrected first-order equations, and then we solve them explicitly in a perturbative approach. We show how

they lead to non-singular modifications of the original Ricci-flat metrics. The perturbative analysis is valid provided that the string scale $\sqrt{\alpha'}$ is small compared with the scale-size L of the Calabi-Yau metric. This scale size is characterised by the size of the bolt at short distance.

Our analysis can easily be extended to include corrections at order higher than ${\alpha'}^3$, provided that one knows the relevant terms in the string effective action. In fact the nature of the possible higher-order terms is restricted severely by the fact that they must satisfy certain universality conditions, and so although not much is known from direct string or sigma-model computations, it is possible to make natural conjectures for the structure of such contributions. This was discussed in detail in [3], where viable corrections at all orders in α' were proposed. Using these terms, we analyse the associated corrections to the various cohomogeneity one Kähler metrics enumerated above.

The paper begins with a discussion of α' corrections in string theory in Section 2. We derive the explicit results for corrections to six-dimensional cohomogeneity one metrics in Section 3, and to eight-dimensional metrics in Section 4. In Section 5 we derive results for the contributions to the Euler numbers for the various Calabi-Yau manifolds that come both from the volume integral of the Euler integrand, and also from the boundary terms that must be included for non-compact manifolds. After the concluding Section 6 where we comment on relations of our work to other α' correction schemes found in the literature, we include appendices summarising results by Chern on the structure of the boundary contributions to the Euler number.

2 α' Corrections in String Theory

At leading order in string loops and α', the effective actions in string theory coincide with type IIA, type IIB or type I supergravities. At higher order, these effective actions are corrected by terms that involve higher derivatives, and higher powers of curvature and field strengths.

Of particular interest are corrections in the type IIA and type IIB string effective actions that are uncovered by studying multi-particle graviton and graviton/dilaton scattering. The leading such corrections in graviton scattering, revealed by four-particle amplitudes, imply the existence of terms in the effective action at order ${\alpha'}^3$, associated with quartic invariants built from the Riemann tensor. The structure of these terms was discovered in early papers on superstrings [4], and a first analysis of their implications for Calabi-Yau

compactifications in string theory was carried out in [5]. The results at first appeared to exhibit puzzling discrepancies in relation to beta-function calculations for sigma-model in Kähler background geometries [6, 19], but a closer study of the quartic-curvature terms from string theory showed that the two approaches were in agreement [7].

The superinvariant structure of the quartic curvature corrections is closely related to that of the ultraviolet counterterms generally anticipated at the three loop order in $D = 4$ supergravity theories or at corresponding lower orders in higher-dimensional theories. These are known for minimal $N = 1$, $D = 4$ supergravity in component form [8] where the full nonlinear structure can be written using off-shell $N = 1$ tensor calculus [9] or in superspace [10]. For the $N = 8$ maximally extended theory (for which no off-shell formalism exists), the structure is known at the quartic order in fields [11, 12]. The $N = 8$ quartic counterterm is the dimensional reduction of the $D = 11$ M-theory quartic correction, which also corresponds to the type IIA string theory one-loop correction [13]. The full supersymmetric nonlinear structure of the $D = 11$ and $D = 10$ quartic corrections is very complicated and remains an unresolved issue, exacerbated by the absence of an off-shell formalism for the maximally supersymmetric theories. A Noether component-field program for supersymmetric construction of the quartic invariants was launched in [14, 15]. Beyond the leading order, however, one has to begin to iteratively correct the supersymmetry transformations as well; the current state of play for this incomplete program is reviewed in [16]. A related issue is the debate on the implications of the quartic corrections to the structure of D-brane backgrounds in Refs [17, 18]. The full component-field construction of the quartic corrections is quite complicated, and we will not be concerned with the general case in the present paper. Instead, we will concentrate on the structure of the corrections as applied to Kähler manifolds without form-field fluxes. This is a more tractable problem, and we shall see that it sheds some light on the general construction.

One of the outcomes of the analysis of α'^3 corrections was that a Ricci-flat Kähler Calabi-Yau metric that solves the internal Einstein equations at leading order ceases to satisfy the equations when the α'^3 terms are present. This was shown in beta-function calculations in [6, 19], and in string-scattering calculations in [7]. The nature of the corrections is relatively mild, in the sense that they imply a distortion of the internal metric under which, for a suitable choice of variables, it remains Kähler, but with the Ricci tensor deformed away from zero in a manner that leaves its cohomology class unchanged. It seems, therefore, that one can treat the corrections as perturbations that smoothly deform the metric away from Ricci flatness, provided that one considers a compactification whose scale size is appreciably

larger than the string length scale $\sqrt{\alpha'}$.

In this paper, we shall focus principally on some explicit calculations exploring the effect of the ${\alpha'}^3$ correction terms at tree level in string theory. It is useful, therefore, to begin by summarising the detailed form of these terms.

In a four-point graviton scattering calculation performed at string tree level in the light-cone gauge, one finds interactions whose covariant description is provided by the contribution

$$\mathcal{L} = -c\,{\alpha'}^3\,e^{-2\phi}\,Y_0 \tag{2.1}$$

in the effective action, where c is a constant,

$$Y_0 \equiv \frac{1}{64}\,t^{i_1\cdots i_8}\,t^{j_1\cdots j_8}\,R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8}\,, \tag{2.2}$$

and the t-tensor is defined by

$$t^{i_1\cdots i_8}\,M_{i_1 i_2}\ldots M_{i_7 i_8} = 24 M_i{}^j\,M_j{}^k\,M_k{}^\ell\,M_\ell{}^i - 6(M_i{}^j\,M_j{}^i)^2 \tag{2.3}$$

for an arbitrary antisymmetric tensor $M_{i_1 i_2}$.

Further information about the quartic-curvature terms comes from considering dilaton/graviton scattering amplitudes, which imply that the total contribution to the effective action must involve a quartic curvature invariant that vanishes in Ricci-flat Kähler backgrounds. This implies that, still in light-cone gauge, the contribution (2.1) is augmented to give

$$\mathcal{L} = -c\,{\alpha'}^3\,e^{-2\phi}\,Y \tag{2.4}$$

where

$$Y \equiv \frac{1}{64}\,\tilde t^{i_1\cdots i_8}\,\tilde t^{j_1\cdots j_8}\,R_{i_1 i_2 j_1 j_2}\cdots R_{i_7 i_8 j_7 j_8} = Y_0 + Y_1 + Y_2 \tag{2.5}$$

and

$$\tilde t^{i_1\cdots i_8} = t^{i_1\cdots i_8} + \tfrac12 \epsilon^{i_1\cdots i_8}\,. \tag{2.6}$$

In (2.5) we are following the notation of [3], in writing the contributions associated with 0, 1 and 2 ϵ-tensor factors as Y_0, Y_1 and Y_2 respectively.

The expression Y in (2.5) can be written in terms of a path integral over $SO(8)$ fermion zero modes, in the form [5]

$$Y = \int d^8\psi_L\,d^8\psi_R\,\exp(R_{ijk\ell}\,\bar\psi_L\,\Gamma^{ij}\,\psi_L\,\bar\psi_R\,\Gamma^{k\ell}\,\psi_R)\,. \tag{2.7}$$

It was shown in [7] that the variation of Y, specialised after variation to a Ricci-flat Kähler background, gives

$$\frac{\delta Y}{\delta g^{ij}} = \nabla_{\hat\imath}\,\nabla_{\hat\jmath}\,S_3\,, \tag{2.8}$$

where here, and in all subsequent formulae, we define

$$\nabla_{\hat{i}} \equiv J_i{}^j \, \nabla_j \,, \tag{2.9}$$

where J_{ij} is the Kähler form, and S_3 is given by

$$S_3 = R_{abcd} \, R^{cdef} \, R_{ef}{}^{ab} - 2R_{acbd} \, R^{cedf} \, R_e{}^a{}_f{}^b \,. \tag{2.10}$$

The expression given in (2.5) does not immediately allow itself to be re-expressed in a ten-dimensionally covariant fashion, since it makes explicit use of the eight-index ϵ-tensor of the transverse eight-dimensional space in the light-cone gauge. The product of two ϵ tensors in Y_2 can be replaced by antisymmetrised products of Kronecker deltas, thus allowing a covariant extension to ten dimensions, but the term Y_1 linear in ϵ admits no direct covariant extension. This problem was studied in [3, 20], and a ten-dimensionally covariant Lagrangian was obtained. Since it is important for our later purposes, we shall review the construction of the ten-dimensionally covariant Lagrangian here, and clarify some of the issues involved.

After straightforward combinatoric manipulations, one finds that the term Y_0, defined in (2.2), is a combination of quartic Riemann-tensor invariants given by:

$$Y_0 = \tfrac{3}{16}(X_0 + 2X_1 + 16X_2 - 16X_3 + 32X_6 - 8X_7) \,. \tag{2.11}$$

Here, we define the quartic Riemann tensor invariants $X_0, \ldots, X_7$, as

$$\begin{aligned}
X_0 &\equiv (R_{abcd} \, R^{abcd})^2 \,, \\
X_1 &\equiv R_{a_1 b_1 a_2 b_2} \, R^{a_2 b_2 a_3 b_3} \, R_{a_3 b_3 a_4 b_4} \, R^{a_4 b_4 a_1 b_1} \,, \\
X_2 &\equiv R_{a_1}{}^{a_2}{}_{b_1}{}^{b_2} \, R_{a_2}{}^{a_3}{}_{b_2}{}^{b_3} \, R_{a_3}{}^{a_4}{}_{b_3}{}^{b_4} \, R_{a_4}{}^{a_1}{}_{b_4}{}^{b_1} \,, \\
X_3 &\equiv R_{a_1 a_2 b_1}{}^{b_2} \, R^{a_1 a_2}{}_{b_2}{}^{b_3} \, R_{a_3 a_4 b_3}{}^{b_4} \, R^{a_3 a_4}{}_{b_4}{}^{b_1} \,, \\
X_4 &\equiv R_{a_1 b_1 b_2 b_4} \, R^{a_1 b_3 a_4 b_4} \, R^{a_2 b_1}{}_{a_3 b_3} \, R_{a_2}{}^{b_2 a_3}{}_{a_4} \,, \\
X_5 &\equiv R_{a_1 a_4 a_3 b_3} \, R^{a_2 b_2 a_3 b_3} \, R^{a_1}{}_{b_1 b_2 b_4} \, R_{a_2}{}^{b_1 a_4 b_4} \,, \\
X_6 &\equiv R_{a_1 a_2 b_1 b_2} \, R^{a_2 a_3 b_2 b_3} \, R_{a_3 a_4 b_4}{}^{b_1} \, R^{a_4 a_1}{}_{b_3}{}^{b_4} \,, \\
X_7 &\equiv R_{a_1 a_2 b_1 b_2} \, R^{a_3 a_4 b_1 b_2} \, R^{a_2}{}_{a_3 b_3 b_4} \, R_{a_4}{}^{a_1 b_3 b_4} \,.
\end{aligned} \tag{2.12}$$

Using the cyclic identity for the Riemann tensor, we have

$$4X_4 + 4X_5 - 4X_6 - X_7 = 0 \,. \tag{2.13}$$

The term Y_2 is proportional to the eight-dimensional Euler integrand E_8, generalised to arbitrary dimension:

$$Y_2 = 384\pi^4 \, E_8 \,, \tag{2.14}$$

with

$$E_8 = \frac{105}{(4\pi)^4}\, R_{a_1 a_2}{}^{[a_1 a_2}\, R_{a_3 a_4}{}^{a_3 a_4}\, R_{a_5 a_6}{}^{a_5 a_6}\, R_{a_7 a_8}{}^{a_7 a_8]}\,. \tag{2.15}$$

We find that the exact expression for Y_2 is given by

$$Y_2 = Y_2^{(0)} + Y_2^{(1)} + Y_2^{(2)}\,, \tag{2.16}$$

where[1]

$$\begin{aligned}
Y_2^{(0)} &\equiv \tfrac{3}{16}(X_0+2X_1+16X_2-16X_3-32X_4+32X_5)\,,\\
Y_2^{(1)} &\equiv 6R^{ab}\,(4R_a{}^{cde}\,R_b{}^f{}_d{}^g\,R_{efcg}+2R_a{}^d{}_b{}^e\,R_{echg}\,R_d{}^{chg}-R_a{}^{cde}\,R_{bcfg}\,R_{de}{}^{fg})\\
&\quad +6R_{abef}\,R_{cd}{}^{ef}\,R^{ac}\,R^{bd}+12R_{aebf}\,R_c{}^e{}_d{}^f\,(R^{ac}\,R^{bd}-R^{ab}\,R^{cd})+12R^{ac}\,R^b{}_c\,R_{adef}\,R_b{}^{def}\\
&\quad -\tfrac{3}{2}R_{abcd}\,R^{abcd}\,R_{ef}\,R^{ef}-24R_{abcd}\,R^{ac}\,R^{be}\,R^d{}_e-6R_{ab}\,R^{bc}\,R_{cd}\,R^{da}\,,\\
Y_2^{(2)} &\equiv R(R_{abcd}\,R^{cdef}\,R_{ef}{}^{ab}-2R_{acbd}\,R^{cedf}\,R_e{}^a{}_f{}^b-R_{abcd}\,R^{abce}\,R^d{}_e\\
&\quad +\tfrac{3}{8}R\,R_{abcd}\,R^{abcd}+6R_{abcd}\,R^{ac}\,R^{bd}+4R_{ab}\,R^{bc}\,R_c{}^a-\tfrac{3}{2}R\,R_{ab}\,R^{ab}+\tfrac{1}{16}R^3)\,.
\end{aligned} \tag{2.17}$$

Of course the terms that are of quadratic or higher order in the Ricci tensor or Ricci scalar are in any case irrelevant here, since even after variation with respect to the metric, their contributions will still vanish at order α'^3 (since we can impose the zero'th-order Ricci-flat Kähler background equations on these corrections that carry an explicit α'^3 factor, after varying to derive the equations of motion). However, terms linear in the Ricci tensor or Ricci scalar *will* contribute to the equations of motion at this order, since the variations of the Ricci terms will themselves give non-vanishing contributions. Thus we just need

$$\begin{aligned}
Y_2^{(1)} &= 6R^{ab}\,(4R_a{}^{cde}\,R_b{}^f{}_d{}^g\,R_{efcg}+2R_a{}^d{}_b{}^e\,R_{echg}\,R_d{}^{chg}-R_a{}^{cde}\,R_{bcfg}\,R_{de}{}^{fg})+\cdots\,,\\
Y_2^{(2)} &= R(R_{abcd}\,R^{cdef}\,R_{ef}{}^{ab}-2R_{acbd}\,R^{cedf}\,R_e{}^a{}_f{}^b)+\cdots\,,
\end{aligned} \tag{2.18}$$

where the ellipses represent terms of quadratic or higher order in the Ricci tensor or scalar, which can be neglected in the present discussion.

Some comment is appropriate on why it is convenient to retain at least some terms involving the Ricci tensor and Ricci scalar. Such terms can of course be adjusted by field redefinitions that cause terms proportional to the leading $(\alpha')^0$ field equations to mix with the quartic corrections. One could decide to use such field redefinitions to eliminate the Ricci tensor and Ricci scalar terms retained in (2.18); one could also proceed further and

[1]Our grouping of terms in Y_2 is as follows. $Y_2^{(0)}$ denotes the terms involving only uncontracted Riemann tensors; $Y_2^{(2)}$ denotes all terms involving at least one Ricci scalar; and $Y_2^{(1)}$ denotes the remainder, namely terms without Ricci scalars and with at least one Ricci tensor.

decide to extract all the Ricci terms from the curvatures in $Y_2^{(0)}$, retaining only Weyl tensors in the place of the curvatures. Different choices of this sort correspond to different choices of field renormalisations. Our choice to retain the terms in (2.18) will have the virtue of allowing us to preserve the Kähler structure of the internal manifold. We will return to the matter of field redefinitions in the Conclusion, in which we compare our results to those of [2].

Another related issue is that of renormalisation-scheme dependence of sigma-model beta functions. As discussed in [20], the coefficients of terms in L that are linear in the Ricci *scalar* (these reside in $Y_2^{(2)}$ in our description) are not determined within sigma-model beta function calculations, since they produce contributions to the equations of motion that are absorbable by redefinitions of the sigma-model scalar fields. To see this more clearly, recall that what one calculates directly from the sigma model are the renormalisation group beta functions, which are taken to give effective field equations for the massless modes when set to zero. The effect of an actively-viewed general coordinate transformation with parameter V_i on the metric is $\delta g_{ij} = \nabla_{(i} V_{j)}$. Moreover, terms in the variation of the effective action proportional to g_{ij} should correspond to the dilaton beta function [21, 22]. Thus, since the variation of $\sqrt{-g} RW$ gives contributions to the gravitational equation of the form $\nabla_i \nabla_j W - g_{ij} \nabla^2 W$ plus terms containing R_{ij}, these contributions can be absorbed into coordinate-transformations of the metric and dilaton. Such terms are *scheme-dependent* from the sigma-model point of view, and can be changed by changes of regularisation and subtraction procedure. Nonetheless, having chosen a specific renormalisation scheme, the coefficients of terms linear in the Ricci scalar do have significance. By contrast, the terms linear in the Ricci *tensor* (residing in $Y_2^{(1)}$ in our description) are not subject to these scheme-dependent ambiguities.

Let us now look at the Y_1 term, which does not admit an obvious generalisation to a fully ten-dimensionally covariant expression whilst maintaining all of the necessary features that its exhibits in special backgrounds. This issue was explored in detail in [3], where it was noted that by a topological property of Kähler manifolds the integral of Y_1 could be replaced by the integral of $-2Y_2^{(2)}$. In this paper, we observe that in an eight-dimensional Ricci-flat Kähler background, Y_1 can in fact be be directly expressed as

$$Y_1 = -2 Y_2^{(0)} \,. \tag{2.19}$$

This can be seen by noting that, viewed as 8-forms, we have

$$Y_1 = 6 \mathrm{tr}\Theta^4 - \tfrac{3}{2} (\mathrm{tr}\Theta^2)^2 \,, \qquad Y_2 = \tfrac{1}{16} \epsilon_{i_1 \cdots i_8}\, \Theta^{i_1 i_2} \cdots \Theta^{i_7 i_8} \,. \tag{2.20}$$

where $\Theta_{ab} = \frac{1}{2} R_{abmn}\, dx^m \wedge dx^n$. Now in a Kähler metric, with Kähler form J_{ij}, we have $\epsilon_{i_1\cdots i_8} = 105 J_{[i_1 i_2} \cdots J_{i_7 i_8]}$. After straightforward combinatoric manipulations, we find that after substituting this into the expression for Y_2, and using $J_{ac} J_{bd}\, \Theta^{cd} = \Theta_{ab}$, then the terms in Y_2 where there is no contraction of the form $J_{ab}\, \Theta^{ab}$ (i.e. the terms where there is no contraction of the Riemann tensor to give a Ricci tensor) are given by

$$-3 \mathrm{tr}\Theta^4 + \tfrac{3}{4} (\mathrm{tr}\Theta^2)^2 \,. \tag{2.21}$$

In other words, we have the result that (2.19) holds in an eight-dimensional Ricci-flat Kähler background. (The effect of Y_1 in correcting special-holonomy backgrounds with flux in M-theory was considered in [23].)

Based on the topological argument mentioned above, it was therefore conjectured in [3] that the appropriate ten-dimensionally covariant generalisation of the light-cone Lagrangian (2.5) at α'^3 order should be given by

$$\mathcal{L} = -c\, \alpha'^3\, e^{-2\phi}\, (Y_0 - Y_2) \,. \tag{2.22}$$

Our observation in Eqn (2.19) lends further support to this proposal. However, it is really only by performing a variation of (2.22) explicitly that one can give a complete verification, since (2.22) was obtained by the potentially hazardous procedure of substituting the Ricci-flat Kähler background condition into the Lagrangian, prior to its variation.

The full Lagrangian, up to this order, should take the form

$$\mathcal{L} = \sqrt{-g}\, e^{-2\phi}\, (R + 4(\partial\phi)^2 - c\, \alpha'^3\, Q) \,, \tag{2.23}$$

where Q is a ten-dimensionally covariant function whose variation $Q_{ij} \equiv \delta Q / \delta g^{ij}$ gives

$$Q_{ij} = \nabla_{\hat{i}}\, \nabla_{\hat{j}}\, S_{(3)} \,, \tag{2.24}$$

when specialised to a Ricci-flat Kähler background. (We are allowed to employ the leading-order Ricci-flat Kähler background equations here, *after* the variation, since there is already an explicit α'^3 factor in the term involving Q.) The dilaton and Einstein equations following from (2.23) give

$$\begin{aligned} &R + 4\Box\phi - 4(\partial\phi)^2 - c\, \alpha'^3\, Q = 0 \,, \\ &R_{ij} + 2\nabla_i \nabla_j\, \phi - c\, \alpha'^3\, Q_{ij} - \tfrac{1}{2}(R + 4\Box\phi - 4(\partial\phi)^2 - c\, \alpha'^3\, Q)\, g_{ij} = 0 \,. \end{aligned} \tag{2.25}$$

Specialising to a Ricci-flat Kähler background and substituting the former into the latter equation in the α'^3 terms as discussed above, gives

$$R_{ij} = \nabla_{\hat{i}} \nabla_{\hat{j}}\, S_3 - 2\nabla_i \nabla_j\, \phi \tag{2.26}$$

Taking the trace of this, substituting into the dilaton equation and neglecting the term $(\partial\phi)^2$, since it would be of order α'^6, gives

$$\Box(2\phi + c\,\alpha'^3\, S_3) = 0\ , \tag{2.27}$$

so we can take [24]

$$\phi = -\tfrac{1}{2}c\,\alpha'^3\, S_3\ . \tag{2.28}$$

Finally, (2.26) then implies that we have

$$R_{ij} = c\,\alpha'^3\,(\nabla_i\nabla_j + \nabla_{\hat\imath}\nabla_{\hat\jmath})\, S_3\,. \tag{2.29}$$

Since this is the desired result, it therefore remains to establish that indeed we can take Q to be given by

$$Q = Y_0 - Y_2\,, \tag{2.30}$$

as proposed in [3] and in agreement with (2.22).

As was noted in [3], and as is evident from (2.11) and (2.17), the Riemann tensor structure appearing in the effective action (2.22) is much simpler than that found in each individual term in the Y's; the full expression in (2.22) is given by

$$\mathcal{L} = e^{-2\phi}\sqrt{-g}\,\Big[R + 4(\partial\phi)^2 - c\,\alpha'^3\,(12(X_6 - X_5) - Y_2^{(1)} - Y_2^{(2)})\Big]\,. \tag{2.31}$$

A convenient way to establish that (2.22) gives the desired form of the equation of motion (2.29) is first to address a slightly different problem, in which one considers the beta function for a pure $N=2$ supersymmetric sigma model without a dilaton. In this case without a dilaton in the model, the vanishing of the beta function at the four-loop level gives rise once again to the condition (2.29). One can ask whether there exists an action for this beta-function equation, and if so, how it relates to the desired string-theory action discussed above. Let us write the beta-function Lagrangian as

$$\mathcal{L}_\sigma = \sqrt{g}\,(R - c\,\alpha'^3\, P)\,. \tag{2.32}$$

A natural ansatz for P is to take

$$P = Y_0 - Y_2^{(0)} + c_1\, Y_2^{(1)} + c_2\, Y_2^{(2)}\,, \tag{2.33}$$

where c_1 and c_2 are constants to be determined. (By contrast, the coefficient of $Y_2^{(0)}$ is determined by the requirement that P should vanish in a Ricci-flat Kähler background.) As we have already mentioned, c_1 and c_2 can be adjusted by field redefinitions. Nonetheless,

if one wants the specific Kähler-preserving structure of the corrected Einstein equation given in (2.29), then c_1 and c_2 are determined uniquely.

This problem of finding an action that produces the sigma-model beta function as its equation of motion was studied in [20]. Here, we shall not perform an explicit variation of (2.32), but rather we shall make use of special cases of cohomogeneity one metrics that admit Ricci-flat Kähler solutions in order to determine the coefficients c_1 and c_2 in (2.33) by requiring consistency with (2.29). We can do this by simply substituting the general cohomogeneity one metric into (2.32) and then obtaining equations of motion by varying the metric functions.[2] The calculations must be performed for eight-dimensional Kähler metrics in order to pin down fully the structure of the Lagrangian. In practice, the calculations are of a sufficient degree of complexity that a computer is helpful.

We have carried out this procedure for many of the metric examples discussed in the later sections of the paper, and we find universal results for the two coefficients c_1 and c_2, namely $c_1 = -1$, $c_2 = -2$. Thus we conclude that the Lagrangian (2.32), with

$$P = Y_0 - Y_2^{(0)} - Y_2^{(1)} - 2Y_2^{(2)}\,, \tag{2.34}$$

gives rise to the $N = 2$ sigma-model beta function.[3] In particular, note that the variation of (2.32) gives

$$R_{ij} - \tfrac{1}{2} R\, g_{ij} - c\,\alpha'^3\, P_{ij} = 0\,, \tag{2.35}$$

where $P_{ij} = \delta P/\delta g^{ij}$, and we have used the fact that P itself vanishes in the Ricci-flat Kähler background. Since $R_{ij} = c\,\alpha'^3\,(\nabla_i \nabla_j + \nabla_{\hat{i}} \nabla_{\hat{j}})\, S_3$, it follows that

$$P_{ij} = \nabla_i \nabla_j\, S_3 + \nabla_{\hat{i}} \nabla_{\hat{j}}\, S_3 - \Box S_3\, g_{ij}\,. \tag{2.36}$$

Having determined the variation of P in (2.34), we can now go back to the tree-level string effective Lagrangian (2.23) including the dilaton, where Q is given by (2.30). Comparing (2.30) and (2.34), we see that

$$Q = P + Y_2^{(2)}\,. \tag{2.37}$$

The relevant terms in $Y_2^{(2)}$ (i.e. those linear in R) are given by $Y_2^{(2)} = R\, S_3 + \cdots$. Using

$$\delta R = (R_{ij} - \nabla_i \nabla_j + g_{ij}\, \Box)\, \delta g^{ij}\,, \tag{2.38}$$

[2] This is a valid procedure provided that one substitutes the most general form of metric invariant under the isometries of the homogeneous level surfaces. Such a shortcut to obtaining a consistent truncation has also been employed in Ref. [17].

[3] In [20], the Lagrangian $\mathcal{L}_{\rm gz} = \sqrt{g}\,[R - c\,\alpha'^3\,(Y_0 - Y_2^{(0)} - Y_2^{(1)} - \frac{4}{3} Y_2^{(2)})]$ is obtained. Evidently, therefore, the scheme employed in [20] differs from ours, for which the coefficients c_1 and c_2 in (2.22) are uniquely defined by the fact that in a Kähler background we have (2.29).

it follows that the variation of $Y_2^{(2)}$, in a Ricci-flat Kähler background, gives

$$\frac{\delta Y_2^{(2)}}{\delta g^{ij}} = -\nabla_i \nabla_j\, S_3 + \Box S_3\, g_{ij}\,. \tag{2.39}$$

Hence, from (2.36), it follows that

$$Q_{ij} = \nabla_{\hat{i}} \nabla_{\hat{j}}\, S_3\,. \tag{2.40}$$

Thus we have verified that taking Q to be given by (2.30) does indeed give the correct string effective Lagrangian.

Finally, we shall make a remark about the structure of the terms proportional to g_{ij} coming from the variation of the Lagrangians we have been considering. These terms are of significance because they should be absent in the metric beta function for the $N = 2$ sigma model. The calculation is slightly subtle, since not only do such terms arise from the obvious source $\delta\sqrt{g}/\delta g^{ij} = -\frac{1}{2}\sqrt{g}\, g_{ij}$, but also from the variation of metrics in R_{ij} and R in $Y_2^{(1)}$ and $Y_2^{(2)}$ respectively. In fact one finds

$$\frac{\delta Y_2^{(1)}}{\delta g^{ij}} = \tfrac{1}{2} \nabla_k \nabla_\ell\, S^{k\ell}\, g_{ij} + \cdots\,, \qquad \frac{\delta Y_2^{(2)}}{\delta g^{ij}} = \Box\, S_3\, g_{ij} + \cdots\,, \tag{2.41}$$

where the ellipses represent the terms not proportional to g_{ij}, and S_{ij} is defined by $Y_2^{(1)} = R^{ij}\, S_{ij}$ (see (2.18)). One can show that in a Ricci-flat background $\nabla_i \nabla_j\, S^{ij} = -2\Box\, S_3$, and hence this explains the g_{ij} term in $\delta P/\delta g^{ij}$ in (2.36), and the absence of the g_{ij} terms in $\delta Q/\delta g^{ij}$ in (2.40).

3 Killing Spinors, Integrability Conditions and Field Equations

As we discussed in Section 2, the equations of motion for the internal Calabi-Yau manifold K_6 in a $(\text{Minkowski})_4 \times K_6$ solution in string theory receive non-vanishing corrections at orders α'^3 and above. These are of the form

$$R_{ab} = (\nabla_a \nabla_b + \nabla_{\hat{a}} \nabla_{\hat{b}})\, S\,, \tag{3.1}$$

where as usual $\nabla_{\hat{a}} \equiv J_a{}^b\, \nabla_b$,

$$S = \sum_{n=3}^{\infty} \alpha'^n\, S_n\,, \tag{3.2}$$

and from this point onwards, we shall choose units where constant c appearing in (2.1) and subsequent formulae in Section 2 is set to unity. Here S_n are certain invariants built

from products of n Riemann tensors. Multiplying (3.1) by $J_k{}^j$, we can recast it in terms of differential forms as

$$\varrho = d\hat{d}\,S\,, \tag{3.3}$$

where the Ricci-form ϱ is defined by

$$\varrho_{ab} \equiv \tfrac{1}{2} R_{abcd}\, J^{cd} = J_b{}^c\, R_{ac}\,, \tag{3.4}$$

and $\hat{d}f \equiv \partial_{\hat{a}} f\, e^a$. Note that we therefore have

$$d = \partial + \bar{\partial}\,, \qquad \hat{d} = -\mathrm{i}\,(\partial - \bar{\partial})\,, \tag{3.5}$$

where ∂ and $\bar{\partial}$ are the holomorphic and anti-holomorphic exterior derivative operators. Thus (3.3) is equivalent to $\varrho = 2\mathrm{i}\,\partial\bar{\partial}S$, showing that the right-hand side can be viewed as a cohomologically trivial $(1,1)$ deformation of the leading-order Ricci-flat condition.

Equations (3.1) or (3.3) define a deformation from Ricci-flatness in which Kählerity is preserved. In fact the solution will also continue to be supersymmetric. It was shown in [24] that a Kähler metric satisfying (3.1) admits Killing spinors that satisfy the modified equation

$$\nabla_a \eta + \tfrac{\mathrm{i}}{2}\,(\partial_{\hat{a}} S)\,\eta = 0\,, \tag{3.6}$$

where $\nabla\eta = d\eta + \frac{1}{4}\omega_{ab}\,\Gamma^{ab}\,\eta$. In fact (3.6) can be written as

$$\nabla\eta + \tfrac{\mathrm{i}}{2}\,(\hat{d}S)\,\eta = 0\,, \tag{3.7}$$

whose integrability condition $(\nabla + \frac{\mathrm{i}}{2}\,(\hat{d}S))^2\eta = 0$ is

$$\tfrac{1}{4}\Theta_{ab}\,\Gamma^{ab}\,\eta + \tfrac{\mathrm{i}}{2}\,\varrho\,\eta = 0\,. \tag{3.8}$$

Writing this in components, $\frac{1}{4}R_{abcd}\,\Gamma^{cd}\,\eta + \frac{\mathrm{i}}{2}\,\varrho_{ab}\,\eta = 0$, and multiplying by Γ^c, it is manifest that the integrability condition is satisfied, by virtue of the holomorphicity condition

$$\Gamma_a\,\eta = -\mathrm{i}\,\Gamma_{\hat{a}}\,\eta\,. \tag{3.9}$$

We shall make use of these observations about the existence of Killing spinors in the following subsections, where we study the effect of the right-hand side of (3.1) in deforming previously-known complete Ricci-flat Kähler metrics. Specifically, from the existence of the Killing spinors we shall be able to derive first-order systems of differential equations for the perturbed metrics, and hence to construct explicit solutions at order ${\alpha'}^3$. We shall apply the technique to three types of six-dimensional Ricci-flat Kähler starting points, namely the resolved conifold, the deformed conifold, and the $\mathbb{R}^2$ bundles over $S^2 \times S^2$ or $\mathbb{CP}^2$.

When we construct fully explicit perturbative solutions, we shall focus first on the term S_3 in (3.2), corresponding to order α'^3. This is the cubic curvature invariant, given in (2.10), that arises in the type IIA and IIB string theories. We shall also consider corrections at higher order in α', namely α'^4 and α'^5. Candidate terms at these, and all higher orders, that satisfy the highly-restrictive *universality conditions* were conjectured in [3].[4]

The universality conditions arise from the fact that, in a sigma-model beta-function calculation, since Kähler or hyper-Kähler target-space background are but specialisations of generic Riemannian backgrounds, it follows that the known special forms of the beta-functions in Kähler or hyper-Kähler backgrounds must be expressible in terms of purely Riemannian quantities. Thus, specifically, the known form of the beta function in a Ricci-flat Kähler background, $\beta_{ab} \sim (\nabla_a \nabla_b + J_a{}^c\, J_b{}^d\, \nabla_c \nabla_d)\, S$, must be expressible in purely Riemannian terms, i.e. without the use of the complex structure $J_a{}^b$. Similarly, since the beta-function is known to vanish in hyper-Kähler backgrounds, the Riemannian expression must have the property of vanishing under this specialisation.

The universality conditions apply similarly to the α' corrected Killing spinor conditions, since these should ultimately have an origin in vanishing gravitino conditions $\delta\psi_m^\alpha = 0$. Indeed, partial results for the corrected gravitino transformation have been derived in $D = 11$ and $D = 10$ supergravities via a Noether supersymmetrisation procedure for the quartic corrections to the action [16]. Conversely, one can use the universality conditions as a guide to finding the structures of correction terms. We observe that the universality properties of S allow the corrected Killing spinor condition (3.6) to be written without the use of complex structures.

There are in fact two different such forms, equivalent when evaluated on Ricci-flat Kähler spaces: one with a Γ_{mnpqrs} structure [24] and one with a Γ_{mn} structure [3]. The six-Γ form is

$$\nabla_i \eta - \tfrac{3}{4} \nabla_s R_{irkl} R^s{}_{tmn} R^{tr}{}_{pq} \Gamma^{klmnpq} \eta = 0\,, \tag{3.10}$$

plus terms that vanish for the leading-order Ricci-flat Kähler solution. The two-Γ form is

$$\nabla_i \eta - 6 \nabla_s R_{ipkl} R^{stln} R_t{}^p{}_n{}^c \Gamma_{ck} \eta = 0\,. \tag{3.11}$$

The equivalence of the two forms for Ricci-flat Kähler spaces is established by dualising

[4]Once one considers corrections beyond $O(\alpha'^5)$ the discussion becomes considerably more complicated, because now one can no longer simply impose the zero'th-order Ricci-flat Kähler background equations on the variations of the correction terms in the Lagrangian. This is because the curvature of the true solution itself has $O(\alpha'^3)$ deviations from its zero'th-order form, and these deviations therefore make $O(\alpha'^6)$ contributions to the variations of the corrections that carry explicit factors of α'^3 and above.

$\Gamma_{i_1\ldots i_6} = -\frac{1}{2}\epsilon_{i_1\ldots i_6 jk}\Gamma_9\Gamma^{jk}$, picking $\Gamma_9\eta = \eta$ and using $J_{[ij}J_{kl}J_{mn}J_{pq]} = \frac{1}{105}\epsilon_{ijklmnpq}$ and $(\Gamma_{ij} + \Gamma_{\hat{i}\hat{j}})\eta = 2\mathrm{i}J_{ij}$, which follows from the Killing spinor holomorphicity condition (3.9). Using the hat-flipping rules to eliminate the complex structures and dropping Ricci tensor terms, one obtains the equivalence of the two forms (3.10,3.11). This equivalence for Ricci-flat Kähler spaces illustrates that the full $D = 10$ or $D = 11$ expression could be a mixture of various forms that become equivalent when evaluated on Ricci-flat Kähler spaces; this impression is borne out by the partial results in [16].

From a geometrical point of view, the two-Γ form (3.11) is noteworthy because it shows that the α' corrections can be viewed as requiring a connection with *torsion* in the Killing spinor connection, with respect to which one simply has $\nabla_i^{\rm corr}\eta = 0$. In order to preserve the Kähler hat-flipping rule $R_{ijk\hat{l}} = -R_{ij\hat{k}l}$, this corrected connection with torsion must remain hermitean, $\nabla_i^{\rm corr} J_{jk} = 0$.

In the next sections, we will use the corrected Killing equation (3.6) to work out explicitly the changes to a set of non-compact Calabi-Yau manifolds.

4 Explicit Non-compact Calabi-Yau Examples in $D = 6$

4.1 Corrections to the resolved conifold

To describe the metric on the resolved and deformed conifolds, it is convenient to introduce the left-invariant 1-forms σ_i and Σ_i for two copies of $SU(2)$. These satisfy

$$d\sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\sigma_j\wedge\sigma_k\,,\qquad d\Sigma_i = -\tfrac{1}{2}\epsilon_{ijk}\,\Sigma_j\wedge\Sigma_k\,. \tag{4.1}$$

We write the metric on the resolved conifold as

$$ds_6^2 = dt^2 + a^2\,(\Sigma_1^2+\Sigma_2^2) + b^2\,(\sigma_1^2+\sigma_2^2) + c^2\,(\Sigma_3-\sigma_3)^2\,, \tag{4.2}$$

and choose the natural vielbein basis

$$e^0 = dt\,,\quad e^1 = a\,\Sigma_1\,,\quad e^2 = a\,\Sigma_2\,,\quad e^3 = b\,\sigma_1\,,\quad e^4 = b\,\sigma_2\,,\quad e^5 = c\,(\Sigma_3-\sigma_3)\,, \tag{4.3}$$

where a, b and c are functions of t. The principal orbits are $T^{1,1} = (S^3\times S^3)/U(1)$, the denominator corresponding to the diagonal $U(1)$ with left-invariant 1-form $(\Sigma_3+\sigma_3)$.

The torsion-free spin connection is easily calculated. It is convenient to present it by giving the Lorentz-covariant exterior derivative $\nabla = d + \frac{1}{4}\omega_{ab}\,\Gamma^{ab}$ that acts on spinors, with vielbein components ∇_a defined by $\nabla = e^a\,\nabla_a$:

$$\nabla_0 = d_0\,,$$

$$
\begin{aligned}
&\nabla_1 = d_1 - \frac{\dot a}{2a}\,\Gamma_{01} - \frac{c}{4a^2}\,\Gamma_{25}\,, \qquad \nabla_2 = d_2 - \frac{\dot a}{2a}\,\Gamma_{02} + \frac{c}{4a^2}\,\Gamma_{15}\,,\\
&\nabla_3 = d_3 - \frac{\dot b}{2b}\,\Gamma_{03} + \frac{c}{4b^2}\,\Gamma_{45}\,, \qquad \nabla_4 = d_4 - \frac{\dot b}{2b}\,\Gamma_{04} - \frac{c}{4b^2}\,\Gamma_{35}\,,\\
&\nabla_5 = d_5 - \frac{\dot c}{2c}\,\Gamma_{05} + \frac{c^2-a^2}{4a^2\,c}\,\Gamma_{12} + \frac{b^2-c^2}{4b^2\,c}\,\Gamma_{34}\,.
\end{aligned} \tag{4.4}
$$

(There are also additional terms $\omega_{12}^{\rm extra} = \omega_{34}^{\rm extra} = -\frac{1}{2}(\Sigma_3 + \sigma_3)$ which lie outside the $S^3 \times S^3)/U(1)$ coset. These project out in the coset construction. See [26] for a further discussion.)

After calculating the curvature from the spin connection, one finds that the Ricci tensor is given by

$$
\begin{aligned}
R_{00} &= -\frac{2\ddot a}{a} - \frac{2\ddot b}{b} - \frac{\dot c}{c}\,,\\
R_{11} &= R_{22} = -\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot a\,\dot c}{a\,c} + \frac{1}{a^2} - \frac{c^2}{2a^4}\,,\\
R_{33} &= R_{44} = -\frac{\ddot b}{b} - \frac{\dot b^2}{b^2} - \frac{\dot b\,\dot c}{b\,c} - \frac{2\dot a\,\dot b}{a\,b} + \frac{1}{b^2} - \frac{c^2}{2b^4}\,,\\
R_{55} &= -\frac{\ddot c}{c} - \frac{2\dot a\,\dot c}{a\,c} - \frac{2\dot b\,\dot c}{b\,c} + \frac{c^2}{2a^4} + \frac{c^2}{2b^4}\,,
\end{aligned} \tag{4.5}
$$

The corrected equations of motion (3.1) for the system are therefore given by

$$
R_{00} = R_{55} = \ddot S + \frac{\dot c}{c}\,\dot S\,, \qquad R_{11} = R_{22} = \frac{2\dot a}{a}\,\dot S\,, \qquad R_{33} = R_{44} = \frac{2\dot b}{b}\,\dot S\,, \tag{4.6}
$$

where the Ricci tensor is given by (4.5).

The system of first-order equations that govern the Ricci-flat resolved conifold itself can easily be derived from (4.4), by requiring the existence of a covariantly-constant spinor, satisfying $\nabla_a\eta = 0$. We can see by inspection that a spinor with constant components, and satisfying the projection conditions

$$
\Gamma_{05}\eta = \Gamma_{12}\eta = -\Gamma_{34}\eta = {\rm i}\,\eta \tag{4.7}
$$

will be covariantly-constant provided that the first-order equations

$$
\dot a = -\frac{c}{2a}\,, \qquad \dot b = -\frac{c}{2b}\,, \qquad \dot c = -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} \tag{4.8}
$$

hold. We can also see that if η is normalised so that $\bar\eta\eta = 1$, then the relation $J_{ab} = -{\rm i}\,\bar\eta\,\Gamma_{ab}\,\eta$ gives the Kähler form,

$$
J = e^0 \wedge e^5 + e^1 \wedge e^2 - e^3 \wedge e^4\,. \tag{4.9}
$$

It is evident from (3.6) that if we now turn on the right-hand side in (3.1), the previous Killing-spinor equations will receive a modification only in the "5" direction, i.e.

$$\begin{aligned}\nabla_a\eta &= 0\,, \qquad 0\le a\le 4\,,\\ \nabla_5\eta - \tfrac{\mathrm{i}}{2}\,\dot S\,\eta &= 0\,.\end{aligned} \tag{4.10}$$

We can immediately see, therefore, that the previous first-order equations for the Ricci-flat case, given in (4.8), will be modified to become

$$\dot a = -\frac{c}{2a}\,,\qquad \dot b = -\frac{c}{2b}\,,\qquad \dot c = -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - c\,\dot S\,. \tag{4.11}$$

It should be emphasised that these are exact equations, valid for any function $S(t)$. In other words, for any S the first-order equations (4.11) imply that that the metric (4.2) will satisfy the modified Einstein equations (4.6). Note that analogous first-order equations were obtained by a different method, and in a different scheme, in [2].

To solve the modified first-order equations, it is convenient to introduce a new radial coordinate ρ, defined by $dt = -c^{-1}\,d\rho$. The first-order equations (4.11) become

$$a' = \frac{1}{2a}\,,\qquad b' = \frac{1}{2b}\,,\qquad c' = \frac{1}{c} - \frac{c}{2a^2} - \frac{c}{2b^2} - c\,S'\,, \tag{4.12}$$

where a prime denotes a derivative with respect to ρ. The functions a and b can be easily solved, giving

$$a^2 = \rho + \ell_1^2\,,\qquad b^2 = \rho + \ell_2^2\,, \tag{4.13}$$

and solving for c we find

$$c^2 = \frac{2}{a^2\,b^2}\,e^{-2S}\int_0^\rho a(x)^2\,b(x)^2\,e^{2S(x)}\,dx\,. \tag{4.14}$$

If S were an externally-specified source term, then this would represent an exact solution to the modified Einstein equations (3.1). It should, however, be emphasised that in the present paper we are taking S to be given by the higher-order corrections to the string effective action, and so S itself is a function of the curvature, and hence a function of a, b, c and their derivatives. In this context, therefore, (4.14) is an integro-differential equation, which in principle determines c.

We can give an explicit solution by linearising the system. Thus we send $S \longrightarrow \varepsilon\,S$, write

$$c = \bar c\,(1 + \varepsilon\, f)\,, \tag{4.15}$$

and now work only to first order in ε. (Note that since the a and b equations in (4.12) do not involve S, their solutions, given in (4.13), remain unchanged by the perturbation.) Substituting (4.15) into (4.12), we find that f can be solved explicitly, to give

$$f = \frac{2\bar{P}}{a^2\, b^2\, \bar{c}^2} - \bar{S}\,. \tag{4.16}$$

Here $\bar{S}$ denotes the curvature invariant appearing in (3.2), evaluated in the unperturbed Ricci-flat metric (i.e. in terms of a and b and the unperturbed metric function $\bar{c}$). The function $\bar{P}$ is defined by

$$\bar{P}(\rho) = \int_0^\rho a(x)^2\, b(x)^2\, \bar{S}(x)\, dx\,. \tag{4.17}$$

If we consider the specific example of the $n = 3$ term in (3.2), we may note that, being the Euler integrand in six dimensions (modulo Ricci tensor terms that vanish in the background), $\sqrt{g}\, S_3$ given in (2.10) is expressible (locally) as a total derivative. In the coordinate gauge we are using here, we therefore have

$$\bar{S}_3 = \frac{1}{a^2\, b^2}\, \frac{d\bar{P}}{d\rho}\,. \tag{4.18}$$

An algebraic computer calculation shows that $\bar{P}$ is given by

$$\bar{P} = k + \frac{18a^2\, b^2\, (a^2+b^2)\, \bar{c}^2 + 12(a^2+b^2)\, \bar{c}^6 - 3(3a^2+b^2)(3b^2+a^2)\, \bar{c}^4}{a^4\, b^4}\,, \tag{4.19}$$

where k is an arbitrary constant.

The Ricci-flat resolved conifold solution [25] is given by setting $\ell_1 = 0$ in (4.13), and evaluating $\bar{c}$ by setting $S = 0$ in (4.14). There is an S^2 bolt at $\rho = 0$, and the metric approaches the cone over $T^{1,1}$ at large ρ. We have

$$a^2 = \rho\,, \qquad b^2 = \rho + \ell^2\,, \qquad \bar{c}^2 = \frac{\rho\,(2\rho + 3\ell^2)}{3(\rho+\ell^2)}\,, \tag{4.20}$$

where we have replaced ℓ_2 by ℓ. We obtain a regular solution for f, which remains finite for $0 \le \rho \le \infty$, by choosing $k = -9$ in (4.19). This gives

$$\begin{aligned} \bar{P} &= \frac{\rho^2\,(\rho+2\ell^2)(7\rho^2 + 21\rho\,\ell^2 + 18\ell^4)}{9(\rho+\ell^2)^5}\,, \\ \bar{S}_3 &= \frac{4\ell^4\,(5\rho^2 + 18\rho\,\ell^2 + 18\ell^4)}{9(\rho+\ell^2)^7}\,. \end{aligned} \tag{4.21}$$

From (4.16) we find

$$f = \frac{2\rho\,(21\rho^4 + 147\rho^3\,\ell^2 + 391\rho^2\,\ell^4 + 471\rho\,\ell^6 + 216\ell^8)}{9(2\rho+3\ell^2)(\rho+\ell^2)^7}\,, \tag{4.22}$$

and so in the linearised level the perturbed solution is given by (4.15), (4.20) and (4.22), where now $\varepsilon = \alpha'^3/\ell^6$.

The function f is non-singular in the entire coordinate range $0 \le \rho \le \infty$. At large ρ we have

$$f = \frac{7}{3\rho^3} - \frac{7\ell^2}{2\rho^2} + \cdots\,, \tag{4.23}$$

while at small ρ we have

$$f = \frac{16\rho}{\ell^8} - \frac{790\rho^2}{9\ell^{10}} + \cdots\,. \tag{4.24}$$

It is clear from this that the regularity of the metric on the S^2 bolt at $\rho = 0$ is unaffected by the perturbation. Of course since we are working only to first-order in perturbations, it is necessary that the parameter of the perturbation expansion be small compared to unity. The relevant dimensionless small parameter is α'/ℓ^2, since ℓ sets the scale size of the bolt where the curvature of the original metric reaches its maximum, at $\rho = 0$. In fact one can see from (4.22) that $|f|$ reaches its maximum at about $\rho \sim 0.23\ell^2$, with $|f|_{\rm max}$ being about $1.2\,\ell^{-6}$. Thus if α'/ℓ^2 is sufficiently small that the first-order perturbation approximation is a good one, then the perturbed solution will be non-singular everywhere.

If one looks at the cone over $T^{1,1}$, corresponding to setting the scale-size ℓ of the resolved conifold to zero, then ostensibly (4.21) implies that $\bar S_3$ vanishes, suggesting that the cone metric itself receives no modification from the corrections at order α'^3. However, this is somewhat misleading since, as can be seen from (4.21), $\bar S_3$ reaches the value $8/\ell^6$ on the bolt in the resolved conifold, and thus it diverges in the limit $\ell \longrightarrow 0$. Thus the assumption that α'/ℓ^2 is everywhere small is violated if one takes the $\ell \longrightarrow 0$ cone limit. If ℓ is set equal to zero, $\bar S_3$ is still divergent at the apex of the cone, now with a delta-function behaviour. Thus again it is strictly-speaking invalid to restrict attention to only the α'^3 corrections in this case.

It is worth remarking that the *local* vanishing of $\bar S_3$ for the six-dimensional cone metric is an immediate consequence of the fact that $\bar S_3$ is the bulk Euler integrand for six-dimensional Ricci-flat metrics and hence $\int_0^\infty \bar S_3\, r^5\, dr$ must be a finite number. (The boundary contributions are discussed in Appendix B.) A *generic* cubic invariant formed from the Riemman tensor will have a c/r^6 power-law behaviour in the six-dimensional cone metric, but the specific invariant in $\bar S_3$ must have $c = 0$ since otherwise $\int_0^\infty \bar S_3\, r^5\, dr$ would be divergent.

In terms of the comoving coordinate t, the functions a, b and c have the following small-distance and large-distance behaviour:

$\underline{t \to 0}$:

$$a = \tfrac{1}{2}t\,\Big(1 - (\frac{1}{72} - \frac{4\alpha'^3}{3\ell^6})\,\frac{t^2}{\ell^2} + \cdots\Big)\,,$$
$$b = \ell\,\Big(1 + \frac{t^2}{8\ell^2} - (\frac{13}{1152} - \frac{\alpha'^3}{3\ell^6})\,\frac{t^4}{\ell^4} + \cdots\Big)\,,$$
$$c = \tfrac{1}{2}t\,\Big(1 - (\frac{1}{18} - \frac{16\alpha'^3}{3\ell^6})\,\frac{t^2}{\ell^2} + \cdots\Big)\,,$$

$\underline{t\to\infty}$:

$$a = \frac{t}{\sqrt{6}}\,\Big(1 - \frac{3\ell^2}{2t^2} + \frac{15\ell^4}{8t^4} - (\frac{207}{80} + \frac{504\alpha'^3}{5\ell^6})\,\frac{\ell^6}{t^6} + \cdots\Big)\,,$$
$$b = \frac{t}{\sqrt{6}}\,\Big(1 + \frac{3\ell^2}{2t^2} + \frac{15\ell^4}{8t^4} - (\frac{657}{80} + \frac{504\alpha'^3}{5\ell^6})\,\frac{\ell^6}{t^6} + \cdots\Big)\,,$$
$$c = \tfrac{1}{3}t\,\Big(1 - \frac{6\ell^4}{t^4} + \tfrac{36}{5}\,(3 + \frac{\alpha'^3}{\ell^6})\,\frac{\ell^6}{\tau^6} + \cdots\Big)\,. \tag{4.25}$$

4.2 Corrections to the deformed conifold

The deformed conifold is a second resolution of the conifold metric, which has the topology of an $\mathbb{R}^3$ bundle over S^3. It can be written in the cohomogeneity-one form

$$ds_6^2 = dt^2 + \tfrac{1}{4}a^2\,[(\sigma_1-\Sigma_1)^2+(\sigma_2+\Sigma_2)^2] + \tfrac{1}{4}b^2\,[(\sigma_2-\Sigma_2)^2+(\sigma_1+\Sigma_1)^2] + \tfrac{1}{4}c^2\,(\Sigma_3-\sigma_3)^2\,, \tag{4.26}$$

for which we choose the vielbein basis

$$e^0 = dt\,,\quad e^1 = \tfrac{1}{2}a\,(\sigma_1 - \Sigma_1)\,,\quad e^2 = -\tfrac{1}{2}a\,(\sigma_2 + \Sigma_2)\,,$$
$$e^3 = \tfrac{1}{2}b\,(\sigma_2 - \Sigma_2)\,,\quad e^4 = \tfrac{1}{2}b\,(\sigma_1 + \Sigma_1)\,,\quad e^5 = \tfrac{1}{2}c\,(\Sigma_3 + \sigma_3)\,. \tag{4.27}$$

The torsion-free spin connection is then summarised in the vielbein components of the Lorentz-covariant exterior derivative $\nabla = d + \frac{1}{4}\omega^{ab}\,\Gamma_{ab}$, which we find to be

$$\begin{aligned}
&\nabla_0 = d_0\,,\\
&\nabla_1 = d_1 - \frac{\dot a}{2a}\,\Gamma_{01} + \tfrac{1}{2}A\,\Gamma_{35}\,,\qquad \nabla_2 = d_2 - \frac{\dot a}{2a}\,\Gamma_{02} + \tfrac{1}{2}A\,\Gamma_{45}\,,\\
&\nabla_3 = d_3 - \frac{\dot b}{2b}\,\Gamma_{03} - \tfrac{1}{2}B\,\Gamma_{15}\,,\qquad \nabla_4 = d_4 - \frac{\dot b}{2b}\,\Gamma_{04} - \tfrac{1}{2}B\,\Gamma_{25}\,,\\
&\nabla_5 = d_5 - \frac{\dot c}{2c}\,\Gamma_{05} + \tfrac{1}{2}C\,(\Gamma_{13} + \Gamma_{24})\,,
\end{aligned} \tag{4.28}$$

where

$$A \equiv \frac{a^2 - b^2 - c^2}{2a\,b\,c}\,,\quad B \equiv \frac{b^2 - a^2 - c^2}{2a\,b\,c}\,,\quad C \equiv \frac{c^2 - a^2 - b^2}{2a\,b\,c}\,. \tag{4.29}$$

There are additional terms $\omega_{12}^{\rm extra} = \omega_{34}^{\rm extra} = -\frac{1}{2}(\Sigma_3+\sigma_3)$ that lie outside the $(S^3\times S^3)/U(1)$ coset, and project to zero, as discussed in [26]. The Ricci curvature is found to be [26]

$$R_{00} = -\frac{2\ddot a}{a} - \frac{2\ddot b}{b} - \frac{\ddot c}{c}\,,$$

$$
\begin{aligned}
R_{11} &= R_{22} = -\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot a\,\dot c}{a\,c} + \frac{a^4 - b^4 - c^4 + 4b^2\,c^2}{2a^2\,b^2\,c^2}\,,\\
R_{33} &= R_{44} = -\frac{\ddot b}{b} - \frac{\dot b^2}{b^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot b\,\dot c}{b\,c} + \frac{b^4 - a^4 - c^4 + 4a^2\,c^2}{2a^2\,b^2\,c^2}\,,\\
R_{55} &= -\frac{\ddot c}{c} - \frac{2\dot a\,\dot b}{a\,b} - \frac{\dot b\,\dot c}{b\,c} + \frac{c^4 - (a^2-b^2)^2}{a^2\,b^2\,c^2}\,.
\end{aligned}
\tag{4.30}
$$

It is evident from (4.28) that a spinor η will be covariantly constant if it has constant components, satisfying the projection conditions

$$
\Gamma_{01}\eta = -\Gamma_{35}\eta\,,\qquad \Gamma_{02}\eta = -\Gamma_{45}\eta\,,
\tag{4.31}
$$

provided that the first-order equations

$$
\dot a = -a\,A\,,\qquad \dot b = -b\,B\,,\qquad \dot c = -2c\,C
\tag{4.32}
$$

hold. These are the first-order equations whose solution yields the Ricci-flat deformed conifold metric. The Kähler form is given by $J_{ab} = -\mathrm{i}\,\bar\eta\,\Gamma_{ab}\,\eta$, which gives

$$
J = -e^0\wedge e^5 + e^1\wedge e^3 + e^2\wedge e^4\,.
\tag{4.33}
$$

If we now consider the corrected equation (3.1), then from (3.6) we see that only the component ∇_5 receives a modification, namely the addition of $\frac{\mathrm{i}}{2}\dot S$, implying that the corrected first-order equations become

$$
\dot a = -a\,A\,,\qquad \dot b = -b\,B\,,\qquad \dot c = -2c\,C - c\,\dot S\,,
\tag{4.34}
$$

where A, B and C are again given by (4.29). If these equations are satisfied, then the metric (4.26) will satisfy the modified Einstein equations (3.1), which are

$$
R_{00} = R_{55} = \ddot S + \frac{\dot c}{c}\,\dot S\,,\qquad R_{11} = R_{22} = \frac{2\dot a}{a}\,\dot S\,,\qquad R_{33} = R_{44} = \frac{2\dot b}{b}\,\dot S\,,
\tag{4.35}
$$

where the Ricci tensor is given by (4.30).

Defining $u = a\,b$ and $v = a/b$, and introducing a new radial variable r such that $dt = c\,dr$, the first-order equations (4.34) become

$$
v' + v^2 - 1 = 0\,,\qquad u' = c^2\,,\qquad \frac{c'}{c} + \frac{c^2}{u} - v - \frac{1}{v} + S' = 0\,,
\tag{4.36}
$$

where a prime denotes a derivative with respect to r. From these we can solve to obtain

$$
v = \coth r\,,\qquad u^3 = \int_0^r e^{-2S}\,\sinh^2 2x\,dx\,,
\tag{4.37}
$$

and hence

$$a^2 = u\, e^S\, \coth r\,, \quad b^2 = u\, e^S\, \tanh r\,, \quad c = \frac{1}{\sqrt{3}\, u}\, e^{-S}\, \sinh 2r\,. \tag{4.38}$$

As in the case of the resolved conifold, if S were an externally-specified function then this would represent an exact solution of the corrected first-order equations, and hence of the corrected Einstein equations (3.1). In our case S is itself an invariant constructed from the Riemann tensor, and so (4.38) is an integro-differential equation.

Working to linear order in the perturbations, we can send $S \longrightarrow \varepsilon\, S$, and write

$$a = \bar{a}\,(1 + \varepsilon\, f)\,, \qquad b = \bar{b}\,(1 + \varepsilon\, f)\,, \qquad c = \bar{c}\,(1 + \varepsilon\, g)\,, \tag{4.39}$$

where the barred variables denote the metric functions in the unperturbed Ricci-flat deformed conifold, and we work to linear order in ε. In fact the Ricci-flat deformed conifold solution is given by

$$\bar{a} = \ell\, R^{\frac{1}{6}}\, (\coth r)^{\frac{1}{2}}\,, \quad \bar{b} = \ell\, R^{\frac{1}{6}}\, (\tanh r)^{\frac{1}{2}}\,, \quad \bar{c} = \tfrac{1}{\sqrt{3}}\, \ell\, R^{-\frac{1}{3}}\, \sinh 2r\,, \tag{4.40}$$

where r is related to t by $dt = \bar{c}\, dr$, and

$$R = \tfrac{1}{8}(\sinh 4r - 4r)\,. \tag{4.41}$$

Substituting (4.39) into (4.38), we now find that the functions f and g are given by

$$f = -\frac{\bar{P}}{\bar{a}^3\, \bar{b}^3}\,, \qquad g = \frac{2\bar{P}}{\bar{a}^3\, \bar{b}^3} - \bar{S}\,. \tag{4.42}$$

Here $\bar{S}$ denotes the curvature invariant appearing in (3.2), evaluated in the unperturbed Ricci-flat metric. The function $\bar{P}$ is then defined as

$$\bar{P}(r) = \int_0^r dx\, \bar{a}(x)^2\, \bar{b}(x)^2\, \bar{c}(x)^2\, \bar{S}(x)\,. \tag{4.43}$$

If we take the special case of the $n = 3$ term in (3.2), then, as we noted earlier, we can express $\sqrt{g}\, S_3$ as a total derivative. In fact $\sqrt{g}$ is nothing but $a^2\, b^2\, c^2$ times angular factors that are independent of the radial variable, and we find that in this case we have

$$P = 96 a^2\, b^2\, c\, C\, [4A^2\, B^2 + (A + B)^2\, C^2]\,, \tag{4.44}$$

where A, B and C are defined in (4.29). Of course one should replace a, b and c by their unperturbed expressions (the barred variables) when substituting into (4.42).

Substituting the explicit expressions (4.40) into the first-order solution, we find that at short distance the perturbed metric functions have the expansion

$$\begin{aligned} a &= 2^{1/3}\, 3^{-1/6}\, \ell\, (1+\tfrac{1152}{25}\varepsilon) \Big[1 + \frac{3(125-55296\varepsilon)\, r^2}{1250} + \cdots\Big]\,, \\ b &= 2^{1/3}\, 3^{-1/6}\, \ell\, (1+\tfrac{1152}{25}\varepsilon) \Big[r - \frac{(125+497664\varepsilon)\, r^3}{3750} + \cdots\Big]\,, \\ c &= 2^{1/3}\, 3^{-1/6}\, \ell\, (1+\tfrac{1152}{25}\varepsilon) \Big[1 + \frac{2(125-124416\varepsilon)\, r^2}{625} + \cdots\Big]\,, \end{aligned} \tag{4.45}$$

where $\varepsilon \equiv {\alpha'}^3/\ell^6$ here. At large distance, we find the perturbed metric functions have the expansion

$$\begin{aligned} a &= 2^{-2/3}\, \ell\, e^{\frac{2}{3}r} \Big[1 + e^{-2r} + \frac{9-24r+5120\varepsilon}{18}\, e^{-4r} + \cdots\Big]\,, \\ b &= 2^{-2/3}\, \ell\, e^{\frac{2}{3}r} \Big[1 + e^{-2r} + \frac{9-24r+5120\varepsilon}{18}\, e^{-4r} + \cdots\Big]\,, \\ c &= 2^{1/3} 3^{-1/2}\, \ell\, e^{\frac{2}{3}r} \Big[1 - \frac{9-24r+5120\varepsilon}{18}\, e^{-4r} + \cdots\Big]\,, \end{aligned} \tag{4.46}$$

We see that the effect of including the perturbation is to keep the metric regular near the S^3 bolt at $r=0$, and provided the scale size ℓ is large enough compared to $\sqrt{\alpha'}$, the metric will be regular for all r. Note, however, that the scale of the metric is modified by a factor $(1+1152{\alpha'}^3/(25\ell^6))$ at short distance. There was no analogous modification to the scale size of the resolved conifold in Section 4.1.

4.3 Corrections to the line bundle over $S^2 \times S^2$

The metric ansatz (4.2) for the resolved conifold also encompasses a different complete Ricci-flat metric, with a different topology. It corresponds to a situation where the principal orbits degenerate to an $S^2\times S^2$ bolt rather than an S^2 bolt. The first-order equations remain the same as in (4.8), with the same modified form (4.11) when the higher-order corrections are turned on. In fact the solutions now correspond simply to taking both ℓ_1 and ℓ_2 to be non-zero in (4.13), so that neither a nor b vanishes as ρ approaches zero. The Ricci-flat solution is then given by [26]

$$a^2 = \rho + \ell_1^2\,, \qquad b^2 = \rho + \ell_2^2\,, \qquad \bar{c}^2 = \frac{\rho\,(2\rho^2 + 3(\ell_1^2+\ell_2^2)\,\rho + 6\ell_1^2\,\ell_2^2)}{3(\rho+\ell_1^2)(\rho+\ell_2^2)}\,. \tag{4.47}$$

The topology of the principal orbits is changed also; one finds that regularity of the metric at $\rho=0$ implies that the period of the $U(1)$ fibre coordinate over $S^2\times S^2$ is now half of its value in the $T^{1,1}$ orbits of the resolved conifold case, and so now the principal orbits are $T^{1,1}/Z_2$. The metric with $\ell_1=\ell_2$ was first given in [27, 28]. For $\ell_1\neq\ell_2$, the metric was given in [29] in a different coordinate system.

The analysis of the corrected solutions for this $\mathbb{R}^2$ bundle over $S^2 \times S^2$ is very similar to that for the resolved conifold in Section 4.1. The only difference in the construction of

the linearly-perturbed solution is that now the constant k in (4.19) must be set to zero, in order to obtain a perturbed solution that is regular at $\rho = 0$. The expressions for $\bar{P}$ and $\bar{S}_3$ are now rather complicated rational functions of ρ, which we shall not present explicitly. They are easily constructed by substituting (4.47) into (4.19) and (4.18). They are both finite everywhere, with asymptotic forms

$$\begin{aligned}\bar{P} &= \frac{88}{9} - \frac{10(\ell_1^2 - \ell_2^2)^2}{9\rho^2} + \cdots\,,\\ \bar{S}_3 &= \frac{20(\ell_1^2 - \ell_2^2)^2}{9\rho^5} - \frac{68(\ell_1^6 + \ell_2^6) - 104\ell_1^2\,\ell_2^2\,(\ell_1^2 + \ell_2^2)}{9\rho^6} + \cdots\end{aligned} \tag{4.48}$$

at large distance, and

$$\begin{aligned}\bar{P} &= \frac{36(\ell_1^2 + \ell_2^2)\,\rho}{\ell_1^2\,\ell_2^2} - \frac{6(15\ell_1^4 + 26\ell_1^2\,\ell_2^2 + 15\ell_2^4)\,\rho^2}{\ell_1^4\,\ell_2^4} + \cdots\,,\\ \bar{S}_3 &= \frac{36(\ell_1^2 + \ell_2^2)}{\ell_1^4\,\ell_2^4} - \frac{24(9\ell_1^4 + 16\ell_1^2\,\ell_2^2 + 9\ell_2^4)\,\rho}{\ell_1^6\,\ell_2^6} + \cdots\end{aligned} \tag{4.49}$$

at short distance. Likewise the expression for f given by (4.16) is quite involved, and so we shall just present its asymptotic forms explicitly here. It is finite everywhere, and at large distance we now find

$$f = \frac{88}{3\rho^3} - \frac{44(\ell_1^2 + \ell_2^2)}{\rho^4} + \cdots\,. \tag{4.50}$$

At small distance, we find

$$f = \frac{12(9\ell_1^4 + 16\ell_1^2\,\ell_2^2 + 9\ell_2^4)\,\rho}{\ell_1^6\,\ell_2^6} - \frac{6(87\ell_1^6 + 211\ell_1^4\,\ell_2^2 + 211\ell_1^2\,\ell_2^4 + 87\ell_2^6)\,\rho^2}{\ell_1^8\,\ell_2^8} + \cdots\,. \tag{4.51}$$

It is evident from these expressions that both ℓ_1 and ℓ_2 must be non-vanishing for these perturbed solutions to be regular. In particular, this means that one cannot simply obtain the modified solution for the resolved conifold by just setting $\ell_1 = 0$ in the modified solution for the $\mathbb{R}^2$ bundle over $S^2 \times S^2$. This is understandable, since we found that it was necessary to choose $k = 0$ rather then $k = -9$ in (4.19) in order to obtain a regular modified solution for the $\mathbb{R}^2$ bundle over $S^2 \times S^2$.

A special case for the $\mathbb{R}^2$ bundle over $S^2 \times S^2$ is when $\ell_1 = \ell_2$, implying that the $S^2 \times S^2$ is itself an Einstein metric. The Ricci-flat solution is then encompassed in the results of [27, 28]. Since the functions in the perturbed solution become much simpler in this case, we shall present them explicitly here. Setting $\ell_1 = \ell_2 = \ell$, we find

$$\bar{P} = \frac{88}{9} - \frac{8\ell^6}{3(\rho + \ell^2)^3} - \frac{64\ell^{18}}{9(\rho + \ell^2)^9}\,,$$

$$
\begin{aligned}
\bar{S}_3 &= \frac{8\ell^6}{(\rho+\ell^2)^6} + \frac{64\ell^{18}}{(\rho+\ell^2)^{12}}\,, \\
f &= \frac{88}{3(\rho+\ell^2)^3} + \frac{40\ell^6}{3(\rho+\ell^2)^6} + \frac{64\ell^{12}}{3(\rho+\ell^2)^9} - \frac{64\ell^{18}}{(\rho+\ell^2)^{12}}\,.
\end{aligned} \tag{4.52}
$$

In the comoving frame, the functions $a = b$ and c have the following short-distance and large-distance behaviours:

$$
\begin{aligned}
&\underline{t \to 0}\;: \\
&\qquad a = \ell\Big(1 + \frac{t^2}{4\ell^2} - (\frac{7}{96} - \frac{34\alpha'^3}{\ell^6})\,\frac{t^4}{\ell^4} + \cdots\Big)\,, \\
&\qquad c = t\Big(1 - (\tfrac{1}{3} - \frac{272\alpha'^3}{\ell^6})\,\frac{t^2}{\ell^2} + \cdots\Big)\,, \\
&\underline{t \to \infty}\;: \\
&\qquad a = \frac{t}{\sqrt{6}}\Big(1 + (\frac{108}{5} - \frac{6336\alpha'^3}{5\ell^6})\,\frac{\ell^6}{t^6} + \cdots\Big)\,, \\
&\qquad c = \tfrac{1}{3}t\Big(1 - \tfrac{144}{5}(3 - \frac{176\alpha'^3}{\ell^6})\,\frac{\ell^6}{t^6} + \cdots\Big)\,,
\end{aligned} \tag{4.53}
$$

4.4 Corrections to the line bundle over $\mathbb{CP}^2$

A final explicit Calabi-Yau example is provided by using the same construction as for the $\mathbb{R}^2$ bundle over the Einstein metric on $S^2 \times S^2$, except that we now replace the $S^2 \times S^2$ by the Fubini-Study metric on $\mathbb{CP}^2$, with the same value for the cosmological constant. The Ricci-flat metric on the $\mathbb{R}^2$ bundle over $\mathbb{CP}^2$ is again a special case of results in [27, 28].

The metric can be written as

$$
ds_6^2 = dt^2 + 6a^2\, d\Sigma_4^2 + c^2\,(dz + A)^2\,, \tag{4.54}
$$

where $d\Sigma_4^2$ is the Fubini-Study metric on $\mathbb{CP}^2$, with its canonical normalisation $R_{ij} = 6g_{ij}$, and dA is proportional to its Kähler form. We have included the factor of 6 in the $d\Sigma_4^2$ term in (4.54) to scale the $\mathbb{CP}^2$ metric to one with $R_{ij} = g_{ij}$, which is the same as we had for the $S^2 \times S^2$ base metric in Section 4.3. The Fubini-Study metric, and the potential A, can be written as [30]

$$
\begin{aligned}
d\Sigma_4^2 &= d\xi^2 + \tfrac{1}{4}\sin^2\xi\,(\sigma_1^2 + \sigma_2^2) + \tfrac{1}{4}\sin^2\xi\,\cos^2\xi\,\sigma_3^2\,, \\
A &= -\tfrac{3}{2}\sin^2\xi\,\sigma_3\,,
\end{aligned} \tag{4.55}
$$

where σ_i are the left-invariant 1-forms of $SU(2)$. It is straightforward to verify that the metric (4.54) is Ricci-flat if the first-order equations

$$
\dot{a} = -\frac{c}{2a}\,, \qquad \dot{c} = -1 + \frac{c^2}{a^2} \tag{4.56}
$$

hold. As expected, these are the same as those for the R^2 bundle over $S^2\times S^2$ with $a=b$. In terms of a new radial variable ρ such that $dt=-c^{-1}\,d\rho$, the solution is again given by

$$a^2=\rho+\ell^2\,,\qquad c^2=\frac{2\rho\,(\rho^2+3\rho\,\ell^2+3\ell^4)}{3(\rho+\ell^2)^2}\,. \tag{4.57}$$

The regularity of the metric at small ρ implies that the $U(1)$ fibre coordinate z should have period $\Delta z=2\pi$, rather than the $\Delta z=6\pi$ that would be required for S^5, and so the principal orbits are S^5/Z_3.

Although the construction of the Ricci-flat Kähler metric is closely parallel to the case where the base is $S^2\times S^2$ rather than $\mathbb{CP}^2$, and the solution involves identical metric functions a and c, we find that the effect of the α'^3 corrections is significantly different. The essential point is that the Riemann tensor for this $\mathbb{R}^2$ bundle over $\mathbb{CP}^2$ is different from that for the $\mathbb{R}^2$ bundle over $S^2\times S^2$, since the Riemann tensors of the four-dimensional bases are different, and hence the functional forms of the Riemann-tensor invariants S differ in the two cases.

The modified first-order equations are of the identical form to (4.12), with $b=a$;

$$a'=\frac{1}{a}\,,\qquad c'=\frac{1}{c}-\frac{c}{a^2}-c\,S'\,, \tag{4.58}$$

and so for the deformed solution we find

$$a^2=\rho+\ell^2\,,\qquad c^2=\frac{2}{a^4}\,e^{-2S}\int_0^\rho a(x)^4\,e^{2S(x)}\,dx\,. \tag{4.59}$$

Perturbatively, we send $S\longrightarrow\varepsilon\,S$, and write $c=\bar c\,(1+\varepsilon\,f)$, finding

$$f=\frac{2\bar P}{a^4\,\bar c^2}-\bar S\,,\qquad \bar P=\int_0^\rho a(x)^4\,\bar S(x)\,dx\,, \tag{4.60}$$

precisely analogously to (4.16). However, now we find that $\bar P_3$ is given by

$$\bar P_3=-\frac{\bar c^2\,(4a^4-6a^2\,\bar c^2+3\bar c^4)}{a^6}\,, \tag{4.61}$$

rather than (4.19). In fact $\bar S_3$ itself now has a much simpler form too, and is given simply by

$$\bar S_3=-\frac{1}{a^4}\,\frac{d\bar P_3}{d\rho}=\frac{8\bar c\,(2a^2-3\bar c^2)^3}{a^8}\,. \tag{4.62}$$

The function f is now given by

$$f=\frac{64}{3(\rho+\ell^2)^3}+\frac{64\ell^6}{3(\rho+\ell^2)^6}+\frac{64\ell^{12}}{3(\rho+\ell^2)^9}-\frac{64\ell^{18}}{(\rho+\ell^2)^{12}}\,. \tag{4.63}$$

It again has the property of vanishing at $\rho=0$ and $\rho=\infty$, but it differs in detail from the result for f in (4.52) for the case of the $\mathbb{R}^2$ bundle over $\mathbb{R}^2$.

In terms of the comoving coordinate t, the function a and c have the following behaviour:

$$\underline{t\to 0}:$$
$$a = \ell\Big(1+\frac{t^2}{4\ell^2}+(-\frac{7}{96}+\frac{32\alpha'^3}{\ell^6})\,\frac{t^4}{\ell^4}+\cdots\Big)\,,$$
$$c = t\Big(1+(-\tfrac{1}{3}+\frac{256\alpha'^3}{\ell^8})\,\frac{t^2}{\ell^2}+\cdots\Big)\,,$$
$$\underline{t\to\infty}:$$
$$a = \frac{t}{\sqrt{6}}\Big(1+(\frac{108}{5}-\frac{4608\alpha'^3}{5\ell^6})\,\frac{\ell^6}{t^6}+\cdots\Big)\,,$$
$$c = \tfrac{1}{3}t\Big(1+(-\frac{432}{5}+\frac{18432\alpha'^3}{\ell^6})\,\frac{\ell^6}{t^6}+\cdots\Big)\,. \qquad (4.64)$$

4.5 Corrections beyond α'^3 order

Candidate correction terms at orders α'^4 and above, consistent with the highly-restrictive conditions of universality, were proposed in [3]. They are given by (3.1) and (3.2), with

$$S_n = R_{r_1\,k_2}{}^{r_2\,k_2}\,R_{r_2\,k_2}{}^{r_3\,k_3}\cdots R_{r_n\,k_n}R^{r_1\,k_1} - 2^{n-2}\,R_{r_1}{}^{r_2}{}_{k_1}{}^{k_2}\,R_{r_2}{}^{r_3}{}_{k_2}{}^{k_3}\cdots R_{r_n}{}^{r_1}{}_{k_n}{}^{k_1}\,. \qquad (4.65)$$

Clearly, up to and including order α'^5, one can still use the Ricci-flat background for calculating S_n, as we did for S_3. Interestingly, all the S_n vanish for the conifold itself, leading us to conjecture that the conifold does not receive any higher-order corrections. The vanishing of S_n for the conifold is non-trivial; it requires the precise relative values of the coefficients of the two terms in (4.65) that were conjectured in [3].

Resolved conifold

For the resolved conifold, we have

$$S_4 = \frac{4\ell^4\,(14\rho^4+86\rho^3\,\ell^2+209\rho^2\,\ell^4+246\rho\,\ell^6+123\ell^8)}{27(\rho+\ell^2)^{10}}\,, \qquad (4.66)$$
$$S_5 = \frac{40\ell^4\,(32\rho^6+277\rho^5\,\ell^2+1032\rho^4\,\ell^4+2140\rho^3\,\ell^6+2645\rho^2\,\ell^8+1890\rho\,\ell^{10}+630\ell^{12})}{243(\rho+\ell^2)^{13}}\,,$$

The corresponding correction to the function c is given by

$$c = \bar{c}\,(1+\alpha'^3\,f+\alpha'^4\,f_4+\alpha'^5\,f_5+\cdots)\,, \qquad (4.67)$$

where f was given in the previous subsection, and f_4 and f_5 are given by

$$f_4 = \frac{2\rho}{189\ell^2\,(\rho+\ell^2)^{10}\,(2\rho+3\ell^2)}\Big(403\rho^7+4030\rho^6\,\ell^2+18135\rho^5\,\ell^4+47576\rho^4\,\ell^6$$
$$+78554\rho^3\,\ell^8+81760\rho^2\,\ell^{10}+49854\rho\,\ell^{12}+13776\ell^{14}\Big)\,,$$

$$
\begin{aligned}
f_5 \;=\;& \frac{2\rho}{18711\ell^4\,(\rho+\ell^2)^{13}\,(2\rho+3\ell^2)}\Big(120047\rho^{10}+1560611\rho^9\,\ell^2+9363666\rho^8\,\ell^4 \\
&+34333442\rho^7\,\ell^6+85661125\rho^6\,\ell^8_1 52736265\rho^5\,\ell^{10}+198033000\rho^4\,\ell^{12} \\
&+185217340\rho^3\,\ell^{14}+120208550\rho^2\,\ell^{16}+49191450\rho\,\ell^{18}+9702000\ell^{20}\Big), \qquad (4.68)
\end{aligned}
$$

Thus we see that the higher-order corrections up to α'^5 all vanish at $\rho = 0$ and $\rho = \infty$. At small ρ, the function c takes the form

$$
c = \bar{c}\,\left(1 + \frac{16\alpha'^3\,(1750\alpha'^2 + 246\alpha'\,\ell + 81\ell^4)}{82\ell^{12}}\,\rho + \mathcal{O}(\rho^2)\right)\,, \qquad (4.69)
$$

whilst at large ρ it takes the form

$$
c = \bar{c}\,\Big(1 + \alpha'^3\,(\tfrac{7}{3} + \frac{403\alpha'}{189\ell^2} + \frac{120047\alpha'^2}{18711\ell^4})\rho^{-3} + \mathcal{O}(\rho^{-4})\Big)\,. \qquad (4.70)
$$

It is interesting to note that the coefficient of a given power of ρ in these expansions receives corrections at each of the higher orders in α'. In the asymptotic region, it is instructive to write the functions a, b and c in terms of a comoving t coordinate, in order to compare the asymptotic deviations of the resolved conifold and the higher-order-corrected resolved conifold from the cone metric itself. These functions are given by

$$
\begin{aligned}
a &= \frac{t}{\sqrt{6}}\,\Big(1 - \frac{3\ell^2}{2t^2} + \frac{15\ell^4}{8t^4} - (\tfrac{207}{80}\ell^6 + \tfrac{504}{5}\alpha'^3 + \tfrac{3224}{35}\alpha'^4\,\ell^{-2} + \tfrac{15366016}{143451}\alpha'^5\,\ell^{-4})\,t^{-6} + \cdots\Big)\,, \\
b &= \frac{t}{\sqrt{6}}\,\Big(1 + \frac{3\ell^2}{2t^2} + \tfrac{15\ell^4}{8t^4} - (\tfrac{657}{80}\ell^6 + \tfrac{504}{5}\alpha'^3 + \tfrac{3224}{35}\alpha'^4\,\ell^{-2} + \tfrac{15366016}{143451}\alpha'^5\,\ell^{-4})\,t^{-6} + \cdots\Big)\,, \\
c \;&=\; \tfrac{1}{3}t\,\Big(1 - \frac{6\ell^4}{t^4} + (\tfrac{108}{5}\ell^6 + \tfrac{2016}{5}\alpha'^3 + \tfrac{12896}{35}\alpha'^4\,\ell^{-2} + \tfrac{2841504}{3465}\alpha'^5\,\ell^{-4})t^{-6} + \cdots\Big)\,, \qquad (4.71)
\end{aligned}
$$

Thus we see that the higher-order corrections modify the asymptotic behaviour in a rather mild fashion, and in particular, they are highly normalisable at large distances. It is interesting to note that purely on dimensional grounds, one might have expected that S_3 could lead to corrections of the form

$$
\alpha'^3\,t^{-6}\,\log t\,, \qquad (4.72)
$$

and in fact had the relative coefficient between the two terms in S_3 given in (2.10) been different, such a term would indeed arise. Thus specific features of the actual higher-order corrections lead to the systematic absence of structures in the series expansions.

<u>Deformed conifold</u>

For the deformed conifold, the explicit expressions for S_4 and S_5 are rather complicated, and we shall not present them in detail. The upshot is that the higher-order corrections

have effects very similar to the α'^3 correction in modifying the small and large distance behaviour. To see this, we note that at small distances, S_3, S_4 and S_5 are similar:

$$\begin{aligned} S_3 &= \tfrac{3456}{125}(-5+24r^2+\cdots)\,, \\ S_4 &= \tfrac{149766^{1/3}}{125}(-5+32r^2+\cdots)\,, \\ S_5 &= \tfrac{17602566^{2/3}}{625}(-1+8r^2+\cdots)\,, \end{aligned} \tag{4.73}$$

whilst at large distances they all vanish. Clearly the leading order correction terms at large distance or at small distance will be determined by the small distance behaviour for the S_n, which in this case are all of the same form. It follows that the corrections to the deformed conifold have the same structure as those for the α'^3 correction, which we have already discussed.

$U(1)$ bundle over $S^2 \times S^2$

For the generic solution, the situation is rather similar, but the structure of the solution is too complex to present here. We shall present only the case with $\ell_1 = \ell_2 \equiv \ell$, i.e, the case considered in [27, 28]. We have

$$\begin{aligned} S_4 &= \frac{64\ell^6}{9(\rho+\ell^2)^7} - \frac{32\ell^{12}}{3(\rho+\ell^2)^{10}} + \frac{2624\ell^{24}}{9(\rho+\ell^2)^{16}}\,, \\ S_5 &= \frac{160\ell^6}{9(\rho+\ell^2)^8} - \frac{320\ell^{12}}{27(\rho+\ell^2)^{11}} + \frac{320\ell^{18}}{9(\rho+\ell^2)^{14}} + \frac{89599\ell^{30}}{27(\rho+\ell^2)^{20}}\,, \end{aligned} \tag{4.74}$$

The corresponding f_4 and f_5 are given by

$$\begin{aligned} f_4 &= \frac{6192}{91r\,\ell^6} + \frac{6192}{91r^4} + \frac{45536\ell^6}{r^7} + \frac{3040\ell^{12}}{39r^{10}} + \frac{2624\ell^{18}}{39r^{13}} - \frac{2624\ell^{24}}{9r^{16}} \\ &\quad -\frac{6292(r+\ell^2)}{91\ell^6\,(r^2+r\,\ell^2+\ell^4)}\,, \\ f_5 &= \frac{112488}{187r^2\,\ell^2} + \frac{112488}{187r^5} + \frac{964520\ell^6}{r^8} + \frac{3065600\ell^{12}}{5049r^{11}} + \frac{84160\ell^{18}}{153r^{14}} \\ &\quad +\frac{89600\ell^{24}}{153r^{17}} - \frac{89600\ell^{30}}{27r^{20}} - \frac{112488}{187\ell^6\,(r^2+r\,\ell^2+\ell^4)}\,. \end{aligned} \tag{4.75}$$

where $r = \rho^2 + \ell^2$. Again, the higher-order corrections vanish for both $\rho = 0$ and $\rho = \infty$. However, the corrections have more importance than in the previous conifold example. In particular, this is the case if we look at the large-distance behaviour. Using the comoving t coordinate, we have

$$\begin{aligned} a &= b = \frac{t}{\sqrt{6}}\Big(1 + (\tfrac{108}{5}\ell^6 - \tfrac{6336}{5}\alpha'^3 - \frac{1337472}{455}\alpha'^4\,\ell^{-2} - \tfrac{24297408\alpha'^5}{935\ell^4})\,t^{-6} + \cdots\Big)\,, \\ c &= \tfrac{1}{3}t\Big(1 + (-\tfrac{432}{5}\ell^6 + \tfrac{25344}{5}\alpha'^3 + \frac{5349888}{455}\alpha'^4\,\ell^{-2} + \tfrac{97189632}{935}\alpha'^5\,\ell^{-4})\,t^{-6} + \cdots\Big) \end{aligned} \tag{4.76}$$

Thus we see that, in this case, the next-to-leading order terms in the conifold expansion are modified by higher-order corrections.

$\underline{U(1) \text{ bundle over } \mathbb{CP}^2}$

In this case, the high-order correction sources S_4 and S_5 are rather simple; they are given by

$$S_4 = \frac{164X^4}{9a^8}\,, \qquad S_5 = \frac{2800X^5}{27a^{10}}\,, \tag{4.77}$$

where $X = (2a^2 - 3\bar{c}^2)/a^2$. It follows straightforwardly that

$$\begin{aligned} f_4 &= \frac{2624}{117}\Big(\frac{3}{\ell^6\, r} + \frac{3}{r^4} + \frac{3\ell^6}{r^7} + \frac{3\ell^{12}}{r^{10}} + \frac{3\ell^{18}}{r^{13}} - \frac{13\ell^{24}}{r^{16}} - \frac{3(r+\ell^2)}{\ell^6\,(r^2+\ell^2\,r+\ell^4)}\Big)\,, \qquad (4.78)\\ f_5 &= \frac{89600}{459}\Big(\frac{3}{r^5} + \frac{3\ell^6}{r^8} + \frac{3\ell^{12}}{r^{11}} + \frac{3\ell^{18}}{r^{14}} + \frac{3\ell^{24}}{r^{17}} - \frac{17\ell^{30}}{r^{20}} + \frac{3(r+\ell^2)}{\ell^4\,r^2\,(r^2+r\,\ell^2+\ell^4)}\Big)\,, \end{aligned}$$

where $r = \rho - \ell^2$. In terms of the comoving coordinate t, the functions a and f have the following behaviour

$\underline{t \to 0}$:

$$\begin{aligned} a &= \ell\Big(1 + \frac{t^2}{4\ell^2} + (-\frac{7}{96} + \frac{32\alpha'^3}{\ell^6} + \frac{5248\alpha'^4}{\ell^8} + \frac{22400\alpha'^5}{\ell^{10}})\,\frac{t^4}{\ell^4} + \cdots\Big)\,,\\ c &= t\Big(1 + (-\tfrac{1}{3} + \frac{256\alpha'^3}{\ell^8} + \frac{41984\alpha'^4}{\ell^8} + \frac{1792000\alpha'^5}{81\ell^{10}})\,\frac{t^2}{\ell^2} + \cdots\Big)\,, \end{aligned}$$

$\underline{t \to \infty}$:

$$\begin{aligned} a &= \frac{t}{\sqrt{6}}\Big(1 + (\frac{108}{5} - \frac{4608\alpha'^3}{5\ell^6} - \frac{188928\alpha'^4}{65\ell^8} - \frac{430080\alpha'^5}{17\ell^{10}})\,\frac{\ell^6}{t^6} + \cdots\Big)\,,\\ c &= \tfrac{1}{3}t\Big(1 + (-\frac{432}{5} + \frac{18432\alpha'^3}{\ell^6} + \frac{755712\alpha'^4}{65\ell^8} + \frac{1720320\alpha'^5}{17\ell^{10}})\,\frac{\ell^6}{t^6} + \cdots\Big) \qquad (4.79) \end{aligned}$$

5 Explicit Non-compact Calabi-Yau Examples in $D = 8$

In this section, we investigate the effects of α'^3 and higher corrections on the various explicit examples of eight-dimensional non-compact Ricci-flat Kähler metrics of cohomogeneity one. These include the cases where the principal orbits are $U(1)$ bundles over $S^2 \times S^2 \times S^2$, $S^2 \times \mathbb{CP}^2$ or $\mathbb{CP}^3$, and the eight-dimensional Stenzel metric, for which the principal orbits are $SO(5)/SO(3)$.

5.1 $U(1)$ bundles over $S^2 \times S^2 \times S^2$

We shall represent these metrics in terms of three sets of left-invariant 1-forms for the group $SU(2)$, denoted by σ_i, Σ_i and ν_i. The eight-dimensional metric is then given by

$$ds_8^2 = dt^2 + a^2\,(\sigma_1^2+\sigma_2^2) + b^2\,(\Sigma_1^2+\Sigma_2^2) + c^2\,(\nu_1^2+\nu_2^2) + g^2\,(\sigma_3+\Sigma_3+\nu_3)^2\,. \qquad (5.1)$$

We introduce the natural vielbein basis

$$\begin{aligned} &e^0 = dt\,, \quad e^1 = a\,\sigma_1\,, \quad e^2 = a\,\sigma_2\,, \quad e^3 = b\,\Sigma_1\,, \quad e^4 = b\,\Sigma_2\,,\\ &e^5 = c\,\nu_1\,, \quad e^6 = c\,\nu_2\,, \quad e^7 = g\,(\sigma_3+\Sigma_3+\nu_3)\,. \end{aligned} \qquad (5.2)$$

Note that the two combinations $L_1 \equiv \sigma_3 - \nu_3$ and $L_2 \equiv \Sigma_3 - \nu_3$ lie outside the coset.

The torsion-free spin connection can be summarised in the expression for the spinor-covariant exterior derivative $\nabla \equiv e^a\,\nabla_a = d + \frac12 \omega_{ab}\,\Gamma^{ab}$,

$$\begin{aligned} &\nabla_0 = d_0\,,\\ &\nabla_1 = d_1 - \frac{\dot a}{2a}\,\Gamma_{01} - \frac{g}{4a^2}\,\Gamma_{27}\,, \qquad \nabla_2 = d_2 - \frac{\dot a}{2a}\,\Gamma_{02} + \frac{g}{4a^2}\,\Gamma_{17}\,,\\ &\nabla_3 = d_3 - \frac{\dot b}{2b}\,\Gamma_{03} - \frac{g}{4b^2}\,\Gamma_{47}\,, \qquad \nabla_4 = d_4 - \frac{\dot b}{2b}\,\Gamma_{04} + \frac{g}{4b^2}\,\Gamma_{37}\,,\\ &\nabla_5 = d_5 - \frac{\dot c}{2c}\,\Gamma_{05} - \frac{g}{4c^2}\,\Gamma_{67}\,, \qquad \nabla_6 = d_6 - \frac{\dot c}{2c}\,\Gamma_{06} + \frac{g}{4c^2}\,\Gamma_{57}\,,\\ &\nabla_7 = d_7 - \frac{\dot g}{2g}\,\Gamma_{07} + \Big(\frac{g}{4a^2} - \frac{1}{6g}\Big)\,\Gamma_{12} + \Big(\frac{g}{4b^2} - \frac{1}{6g}\Big)\,\Gamma_{34} + \Big(\frac{g}{4c^2} - \frac{1}{6g}\Big)\,\Gamma_{56}\,. \end{aligned} \qquad (5.3)$$

(There are also extra contributions $\omega_{12}^{\rm extra} = \frac13(L_2 - 2L_1)$, $\omega_{34}^{\rm extra} = \frac13(L_1 - 2L_2)$, $\omega_{56}^{\rm extra} = \frac13(L_1 + L_2)$, involving the two directions outside the coset; these project out as discussed in [26].)

It is easily seen that we can find two Killing spinors η satisfying $\nabla\,\eta = 0$ which imply the first-order bosonic equations

$$2a\,\dot a = 2b\,\dot b = 2c\,\dot c = g\,, \qquad \dot g = 1 - \tfrac12 g^2\,\Big(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\Big)\,. \qquad (5.4)$$

The spinors η have constant components, and satisfy the projection conditions

$$\Gamma_{12}\,\eta = \Gamma_{34}\,\eta = \Gamma_{56}\,\eta = -\Gamma_{07}\,\eta\,. \qquad (5.5)$$

The Kähler form can be written as $J_{ab} = -{\rm i}\,\bar\eta\,\Gamma_{ab}\,\eta$, and is given by

$$J = e^1\wedge e^2 + e^3\wedge e^4 + e^5\wedge e^6 - e^0\wedge e^7\,. \qquad (5.6)$$

The first-order equations that arise as integrability conditions for the modified Killing-spinor equation (3.6) are easily seen to be given by

$$\dot a = \frac{g}{2a}\,,\qquad \dot b = \frac{g}{2b}\,,\qquad \dot c = \frac{g}{2c}\,,\qquad \dot g = 1 - \frac{g^2}{2a^2} - \frac{g^2}{2b^2} - \frac{g^2}{2c^2} - g\,\dot S\,. \tag{5.7}$$

As in the previous examples, one can easily verify that if these equations are satisfied then the Einstein equations $R_{ab} = \nabla_a\nabla_b S + \nabla_{\hat a}\nabla_{\hat b} S$ are satisfied, where as usual $\nabla_{\hat a} \equiv J_a{}^b\,\nabla_b$. These second-order equations are

$$R_{00} = R_{77} = \ddot S + \frac{\dot g}{g}\,\dot S\,,\quad R_{11} = R_{22} = \frac{2\dot a}{a}\,\dot S\,,\quad R_{33} = R_{44} = \frac{2\dot b}{b}\,\dot S\,,\quad R_{55} = R_{66} = \frac{2\dot c}{c}\,\dot S\,, \tag{5.8}$$

where the Ricci tensor is given by

$$\begin{aligned}
R_{00} &= -\frac{2\ddot a}{a} - \frac{2\ddot b}{b} - \frac{2\ddot c}{c} - \frac{\ddot g}{g}\,,\\
R_{11} &= R_{22} = -\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{2\dot a\,\dot c}{a\,c} - \frac{\dot a\,\dot g}{a\,g} + \frac{2a^2 - g^2}{2a^4}\,,\\
R_{33} &= R_{44} = -\frac{\ddot b}{b} - \frac{\dot b^2}{b^2} - \frac{2\dot a\,\dot b}{a\,b} - \frac{2\dot b\,\dot c}{b\,c} - \frac{\dot b\,\dot g}{b\,g} + \frac{2b^2 - g^2}{2b^4}\,,\\
R_{55} &= R_{66} = -\frac{\ddot c}{c} - \frac{\dot c^2}{c^2} - \frac{2\dot a\,\dot c}{a\,c} - \frac{2\dot b\,\dot c}{b\,c} - \frac{\dot c\,\dot g}{c\,g} + \frac{2c^2 - g^2}{2c^4}\,,\\
R_{77} &= -\frac{\ddot g}{g} - \frac{2\dot g}{g}\,(\frac{\dot a}{a} + \frac{\dot b}{b} + \frac{\dot c}{c}) + \tfrac12 g^2\,(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4})\,.
\end{aligned} \tag{5.9}$$

Introducing a new radial variable ρ such that $d\rho = g\,dt$, it is easily seen that the solution to the modified first-order equations (5.7) is given by

$$\begin{aligned}
&a^2 = \rho + \ell_1^2\,,\qquad b^2 = \rho + \ell_2^2\,,\qquad c^2 = \rho + \ell_3^2\,,\\
&g^2 = \frac{2}{a^2\,b^2\,c^2}\,e^{-2S}\,\int_0^\rho a^2(x)\,b^2(x)\,c^2(x)\,e^{2S(x)}\,dx\,.
\end{aligned} \tag{5.10}$$

As in our previous examples, this result is exact, and it is explicit (up to quadratures) if S is a given externally-specified function.

Our present interest is in the case where S is some higher-order correction term coming from string theory, as in the discussion of the previous sections. We again therefore make a linearised approximation, in which the quantity S is expressed in terms of the background Riemann tensors of the original Ricci-flat equations. Sending $S \longrightarrow \varepsilon\,S$, and writing $g = \bar g\,(1 + \varepsilon\,f)$, where $\bar f$ is the expression for f at zero'th order in ε, we therefore find, up to linearised order, that the metric functions are given by

$$\begin{aligned}
&a^2 = \rho + \ell_1^2\,,\qquad b^2 = \rho + \ell_2^2\,,\qquad c^2 = \rho + \ell_3^2\,,\\
&f_3 = \frac{2\bar P}{a^2\,b^2\,c^2\,\bar f^2} - \bar S\,,
\end{aligned} \tag{5.11}$$

where

$$P_3 \equiv \int_0^\rho a^2(x)\, b^2(x)\, c^2(x)\, S(x)\, dx \qquad (5.12)$$

and the quantities $\bar{P}$ and $\bar{S}$ are evaluated in the zero'th-order Ricci-flat background.

The general structure with ℓ_i not equal is rather complicated to present. We shall give explicit results only for the case with $\ell_i = \ell$. First let us consider the simplest case with $\ell = 0$, where the metric is just the cone over the $U(1)$ bundle over $S^2 \times S^2 \times S^2$. Unlike the six-dimensional conifold, where S_3 vanishes locally (because, as we discussed, it is the six-dimensional Euler integrand), here it is non-vanishing and is given by $S_3 = 3/\rho^3$. It follows that the perturbation function f_3 is given by $f_3 = 9\epsilon/\rho^3$. This raises the possibility that the string higher-order corrections might have the effect of resolving the singularity of the cone metric itself.

For $\ell \neq 0$, the perturbation function f_3 is given by

$$f_3 = \frac{105}{r^3} + \frac{153\ell^8}{2r^7} + \frac{90\ell^{16}}{r^{11}} - \frac{495\ell^{24}}{2r^{15}} - \frac{48}{\ell^4\,(r+\ell^2)} + \frac{48(r-\ell^2)}{\ell^4\,(r^2+\ell^4)}\,, \qquad (5.13)$$

where $r = \rho - \ell^2$. In the comoving t coordinate, the asymptotic behaviour of the metric functions is

$\underline{t \rightarrow 0}$:

$$a = \ell + \frac{t^2}{4\ell} - (\frac{3}{32\ell^3} - \frac{159\epsilon}{\ell^9})\, t^4 + \cdots\,, \qquad g = t - (\frac{1}{2\ell^2} - \frac{1272\epsilon}{\ell^8})\, t^3 + \cdots\,,$$

$\underline{t \rightarrow \infty}$:

$$a = \frac{t}{2\sqrt{2}}\Big(1 - \frac{4608\epsilon}{5t^6} + \frac{2048(\ell^8 - 192\ell^2\,\epsilon^2)}{7t^8} + \cdots\Big)\,,$$

$$g = \tfrac{1}{4}t\,\Big(1 + \frac{18432\epsilon}{5t^6} - \frac{12288(\ell^8 - 192\ell^2\,\epsilon)}{7t^8} + \cdots\Big)\,. \qquad (5.14)$$

Note that at large distances, the higher-order corrections modify terms occurring before the next-to-leading order terms of the uncorrected expansion. This is because, unlike in six dimensions, the integral of S_3 diverges at large distance in eight dimensions.

5.2 $U(1)$ bundle over $S^2 \times \mathbb{CP}^2$

When $b = a$, we can replace the $S^2 \times S^2$ with the metric for $\mathbb{CP}^2$. The metric ansatz now becomes

$$ds^2 = dt^2 + a^2 d\Sigma_4^2 + c^2\, d\Omega_2^2 + f^2\,(dz - \tfrac{3}{2}\sin^2\xi\,\sigma_3 + A)^2\,, \qquad (5.15)$$

where $dA = \Omega_{(2)}$, and $d\Sigma_4^2$ is given by (4.55). At the zero'th order, the solutions for a, c and f are identical to that of the previous case. However, since the Riemann tensor for $S^2 \times S^2$

is different from that for $\mathbb{CP}^2$, it follows that the α'^3 correction term S_3 is different in this case from the $S^2 \times S^2 \times S^2$ case. For simplicity, we shall only present the result when the constants are chosen so that $a = c = \sqrt{\rho + \ell^2}$. For the cone metric (i.e. $\ell = 0$) we now find $S_3 = 1/r^3$ instead of $3/r^3$ for the $S^2 \times S^2 \times S^2$ case. It follows that there are differences in the higher-order corrections, but they are qualitatively the same.

For $\ell \neq 0$, we find that

$$f_3 = \frac{99}{r^3} + \frac{165\ell^8}{2r^7} + \frac{90\ell^{16}}{r^{11}} - \frac{495\ell^{24}}{2r^{15}} - \frac{48}{\ell^4\,(r+\ell^2)} + \frac{48(r-\ell^2)}{\ell\,(r^2+\ell^4)}\,. \tag{5.16}$$

where again $r = \rho - \ell^2$. Thus structurally, the correction terms are the same as those for the $U(1)$ bundle over $S^2 \times S^2 \times S^2$, but the detailed coefficients are rather different, In the comoving frame, at small distance t, a and f are given by

$\underline{t \to 0}$:

$$a = \ell + \frac{t^2}{4\ell} - (\frac{3}{32\ell^3} - \frac{157\epsilon}{\ell^9})\,t^4 + \cdots\,, \qquad g = t - (\frac{1}{2\ell^2} - \frac{1256\epsilon}{\ell^8})\,t^3 + \cdots\,,$$

$\underline{t \to \infty}$:

$$\begin{aligned} a &= \frac{t}{2\sqrt{2}}\Big(1 - \frac{1536\epsilon}{5t^6} + \frac{2048(\ell^8 - 192\ell^2\,\epsilon^2)}{7t^8} + \cdots\Big)\,,\\ g &= \tfrac14 t\Big(1 + \frac{6144\epsilon}{5t^6} - \frac{12288(\ell^8 - 192\ell^2\,\epsilon)}{7t^8} + \cdots\Big)\,, \end{aligned} \tag{5.17}$$

5.3 $U(1)$ bundle over $\mathbb{CP}^3$

When $a = b = c$, we can replace $S^2 \times S^2 \times S^2$ with $\mathbb{CP}^3$. There are two convenient ways to write the $\mathbb{CP}^3$ metric. One way is to use the recursive expression for the Fubini-Study metric $d\Sigma_{2n}^2$ on $\mathbb{CP}^n$ in terms of the Fubini-Study metric $d\Sigma_{2n-2}^2$ on $\mathbb{CP}^{n-1}$, which was derived in [31]:

$$d\Sigma_{2n}^2 = d\alpha^2 + \sin^2\alpha\, d\Sigma_{2n-2}^2 + \sin^2\alpha\,\cos^2\alpha\,(d\tau + B)^2\,, \tag{5.18}$$

where $dB = 2J_{n-1}$, and J_{n-1} is the Kähler form of $\mathbb{CP}^{n-1}$. For each n, $d\Sigma_n^2$ denotes the canonically-normalised Fubini-Study metric, with $R_{ij} = 2(n+1)\,g_{ij}$. Thus before taking the $O(\alpha')$ corrections into account, first-order equations for the metric

$$ds_8^2 = dt^2 + 8a^2\, d\Sigma_6^2 + g^2\,(dz + A)^2 \tag{5.19}$$

will be the same as those for the $S^2 \times S^2 \times S^2$ base given in Section 5.1, with $a = b = c$. The Kähler form for the $\mathbb{CP}^n$ metric (5.18) is given by $J_n = \frac12 dA$, where $A = \sin^2\alpha\,(d\tau + B)$ [31]. Using (5.18) the metric on $\mathbb{CP}^3$ can be written in terms of the $\mathbb{CP}^2$ metric (4.55), with $A = -\frac14\sin^2\xi\,\sigma_3$.

An alternative construction for the $\mathbb{CP}^n$ metrics can be given by introducing left-invariant 1-forms $L_A{}^B$ for the group $SU(n+1)$, where $0 \le A \le n$, $L_A{}^A = 0$, and $dL_A{}^B = \mathrm{i}\, L_A{}^C \wedge L_C{}^B$. Writing $A = (0, i)$, where $1 \le i \le n$, we can obtain a vielbein for the coset $\mathbb{CP}^n = SU(n+1)/U(n)$ by taking just the subset $L_0{}^i$ and $L_i{}^0$ of the left-invariant 1-forms, i.e. by modding out by the $SU(n)$ 1-forms $L_i{}^j$ and $U(1)$ 1-form $L_0{}^0$. In a real basis, we can define

$$e^i = \tfrac{1}{2}(L_0{}^i + L_i{}^0),\,, \qquad e^{\tilde{i}} = \tfrac{1}{2\mathrm{i}}\,(L_0{}^i - L_i{}^0)\,. \tag{5.20}$$

The spin connection and curvature 2-forms for $\mathbb{CP}^n$ are therefore given by

$$\begin{aligned} &\omega_{ij} = \omega_{\tilde{i}\tilde{j}} = \tfrac{1}{2\mathrm{i}}\,(L_i{}^j - L_j{}^i)\,, \qquad \omega_{i\tilde{j}} = L_0{}^0\,\delta_{ij} - \tfrac{1}{2}(L_i{}^j + L_j{}^i)\,,\\ &\Theta_{ij} = \Theta_{\tilde{i}\tilde{j}} = e^i \wedge e^j + e^{\tilde{i}} \wedge e^{\tilde{j}}\,, \qquad \Theta_{i\tilde{j}} = e^i \wedge e^{\tilde{j}} + e^j \wedge e^{\tilde{i}} + 2e^k \wedge e^{\tilde{k}}\,\delta_{ij}\,, \end{aligned} \tag{5.21}$$

and the Kähler form is

$$J = e^i \wedge e^{\tilde{i}}\,. \tag{5.22}$$

Thus we see that $R_{ij} = 2(n+1)\,\delta_{ij}$, and hence the metric $d\Sigma_{2n}^2 = e^i\,e^i + e^{\tilde{i}}\,e^{\tilde{i}}$ is the canonically-normalised Fubini-Study metric on $\mathbb{CP}^n$.

Using either of the above constructions, it is a straightforward matter to calculate the curvature for the metric (5.19), and hence to show that the cubic Riemann tensor invariant is given in this case by

$$S_3 = \frac{495(a^2 - 2g^2)^3}{2a^{12}}\,. \tag{5.23}$$

Clearly, the cone of the $U(1)$ bundle over $\mathbb{CP}^3$, corresponding to $a = \sqrt{2}\,g$, is locally Euclidean since the principal orbits are locally the round S^7, and hence locally the curvature and all higher-order corrections vanish. If $\ell \ne 0$, we have

$$f_3 = \frac{90}{r^3} + \frac{90\ell^8}{r^7} + \frac{90\ell^{16}}{r^{11}} - \frac{495\ell^{24}}{2r^{15}} - \frac{45}{\ell^4\,(r+\ell^2)} - \frac{45(r-\ell^2)}{\ell^4\,(r^2+\ell^4)}\,, \tag{5.24}$$

where $r = \rho - \ell^2$. Since now S_3 is normalisable, the correction is very different from the previous ones. In the comoving frame, a and g have the asymptotic forms

$\underline{t \to 0}$:

$$a = \ell + \frac{t^2}{4\ell} - (\frac{3}{32\ell^3} - \frac{2475\epsilon}{16\ell^9})\,t^4 + \cdots\,, \qquad g = t - (\frac{1}{2\ell^2} - \frac{2475\epsilon}{2\ell^8})\,t^3 + \cdots\,,$$

$\underline{t \to \infty}$:

$$\begin{aligned} a &= \frac{t}{2\sqrt{2}}\Big(1 + \frac{2048(\ell^8 - 180\ell^2\,\epsilon^2)}{7t^8} + \cdots\Big)\,,\\ g &= \tfrac{1}{4}t\Big(1 - \frac{12288(\ell^8 + 30\ell^2\,\epsilon)}{7t^8} + \cdots\Big)\,. \end{aligned} \tag{5.25}$$

5.4 Stenzel metrics; $SO(5)/SO(3)$ orbits

We shall closely follow the notation of [26] for writing the cohomogeneity one metrics with $SO(n+2)/SO(n)$ principal orbits:

$$ds^2 = dt^2 + a^2\,\sigma_i^2 + b^2\,\tilde\sigma_i^2 + c^2\,\nu^2\,, \tag{5.26}$$

where $3 \le i \le n+2$,

$$\sigma_i \equiv L_{1i}\,, \qquad \tilde\sigma_i \equiv L_{2i}\,, \qquad \nu \equiv L_{12}\,, \tag{5.27}$$

and the L_{AB} with $1 \le A \le n$ are the left-invariant 1-forms of the group $SO(n+2)$, satisfying $dL_{AB} = L_{AC} \wedge L_{CB}$, with $L_{AB} = -L_{BA}$. We choose the natural orthonormal basis

$$e^0 = dt\,, \qquad e^i = a\,\sigma_i\,, \qquad e^{\tilde i} = b\,\tilde\sigma_i\,, \qquad e^{\tilde 0} = c\,\nu\,. \tag{5.28}$$

We take $\bar e^0 \equiv \nu$ and $\bar e^i \equiv \sigma_i$ as a vielbein basis $\bar e^a$ for the sphere $S^{n+1} = SO(n+2)/SO(n+1)$. A simple calculation shows that the torsion-free spin connection is given by $\bar\omega_{0i} = -\tilde\sigma_i$, $\omega_{ij} = -L_{ij}$, and hence that the curvature 2-forms for the metric $d\bar s^2 \equiv \bar e^a\,\bar e^a = \sigma_i^2 + \nu^2$ are given by $\bar\Theta_{ab} = e^a \wedge e^b$. This proves that $d\bar s_n^2$ is the metric on the *unit* $(n+1)$-sphere. Thus the $SO(n+2)/SO(n)$ principal orbits in (5.26) can be viewed as S^n fibres over a (squashed) S^{n+1} base, with $\tilde\sigma_i$ being 1-forms on the S^n fibres.[5]

Calculating the torsion-free spin connection for (5.28), one finds that the spinor covariant exterior derivative is given by

$$\begin{aligned}
\nabla_0 &= d_0\,,\\
\nabla_i &= d_i - \frac{\dot a}{2a}\,\Gamma_{0i} - \tfrac12 A\,\Gamma_{\tilde 0\tilde i}\\
\nabla_{\tilde i} &= d_{\tilde i} - \frac{\dot b}{2b}\,\Gamma_{0\tilde i}\tfrac12 B\,\Gamma_{\tilde 0 i}\\
\nabla_{\tilde 0} &= d_{\tilde 0} - \frac{\dot c}{2c}\,\Gamma_{0\tilde 0} + \tfrac12 C\,\Gamma_{i\tilde i}\,,
\end{aligned} \tag{5.29}$$

where

$$A \equiv \frac{a^2 - b^2 - c^2}{2a\,b\,c}\,, \quad B \equiv \frac{b^2 - a^2 - c^2}{2a\,b\,c}\,, \quad C \equiv \frac{c^2 - a^2 - b^2}{2a\,b\,c}\,. \tag{5.30}$$

(There are also additional terms $\omega_{ij}^{\rm extra} = \omega_{\tilde i\tilde j}^{\rm extra} = -L_{ij}$ that lie outside the coset, and that project to zero [26].)

It is evident from these first-order equations that there is a solution whose short-distance behaviour (near $t=0$) takes the form

$$ds^2 = dt^2 + t^2\,\tilde\sigma_i^2 + a_0^2\,(\sigma_i^2 + \nu^2)\,. \tag{5.31}$$

[5] The roles of the σ_i and the $\tilde\sigma_i$ are symmetrical in this description, and they could be interchanged.

This is precisely the short-distance behaviour of the Stenzel metrics [26], which are complete and non-singular. It is clear from (5.31) that the metric $\tilde\sigma_i^2$ on the S^n fibres must describe a sphere of *unit* radius, in view of the regularity at $t=0$. Thus we can conclude that the principal $SO(n+2)/SO(n)$ orbits in the Stenzel metrics have a volume given by

$$\mathrm{Vol}(SO(n+2)/SO(n)) = \int \nu \wedge \prod_i \sigma_i \wedge \prod_i \tilde\sigma_i = \mathrm{Vol}(S^{n+1})\,\mathrm{Vol}(S^n)\,. \tag{5.32}$$

Since the volume of the unit n-sphere is $\mathrm{Vol}(S^n) = 2\pi^{(n+1)/2}/\Gamma((n+1)/2)$, it follows that the volumes of the principal orbits in the Stenzel metrics are given by

$$\mathrm{Vol}(SO(n+2)/SO(n)) = \frac{2^{n+2}\,\pi^{n+1}}{n!}\,. \tag{5.33}$$

We shall make use of this result later, when calculating the contributions of the volume and boundary terms in the expression for the Euler number.

Specialising to $D=8$ (i.e. $n=3$), the Ricci tensor is given by [26]

$$\begin{aligned}
R_{00} &= -\frac{3\ddot a}{a} - \frac{3\ddot b}{b} - \frac{\ddot c}{c}\,,\\
R_{11} &= R_{22} = R_{33} = -\frac{\ddot a}{a} - \frac{2\dot a^2}{a^2} - \frac{3\dot a\,\dot b}{a\,b} - \frac{\dot a\,\dot c}{a\,c} + \frac{a^4 - b^4 - c^4 + 6b^2\,c^2}{2a^2\,b^2\,c^2}\,,\\
R_{44} &= R_{55} = R_{66} = -\frac{\ddot b}{b} - \frac{2\dot b^2}{b^2} - \frac{3\dot a\,\dot b}{a\,b} - \frac{\dot b\,\dot c}{b\,c} + \frac{b^4 - a^4 - c^4 + 6a^2\,c^2}{2a^2\,b^2\,c^2}\,,\\
R_{77} &= -\frac{\ddot c}{c} - \frac{3\dot a\,\dot c}{a\,c} - \frac{3\dot b\,\dot c}{b\,c} + \frac{3(c^2 - (a^2-b^2)^2)}{2a^2\,b^2\,c^2}\,.
\end{aligned} \tag{5.34}$$

As in the previous examples, we can read off from the covariant exterior derivative (5.29) the first-order integrability conditions for the existence of covariantly-constant spinors $\nabla\eta = 0$, giving

$$\dot a = -a\,A\,,\qquad \dot b = -b\,B\,,\qquad \dot c = -3c\,C\,, \tag{5.35}$$

where the spinors have constant components and satisfy the projection conditions

$$\Gamma_{0i}\,\eta + \Gamma_{\tilde 0\tilde i}\,\eta = 0\,. \tag{5.36}$$

The Kähler form can be written as $J_{ab} = -\mathrm{i}\,\bar\eta\Gamma_{ab}\,\eta$, giving

$$J = -e^0\wedge e^{\tilde 0} + e^i \wedge e^{\tilde i}\,. \tag{5.37}$$

The integrability conditions for the modified Killing spinor equation (3.6) are then easily seen to be

$$\dot a = -a\,A\,,\qquad \dot b = -b\,B\,,\qquad \dot c = -3c\,C - c\,\dot S\,. \tag{5.38}$$

As in the previous cases, one can verify that if these equations are satisfied, then the metric satisfies the modified Einstein equations $R_{ab} = \nabla_a \nabla_b S + \nabla_{\hat{a}} \nabla_{\hat{b}} S$. Explicitly, these equations are

$$R_{00} = R_{\tilde{0}\tilde{0}} = \ddot{S} + \frac{\dot{c}}{c}\,\dot{S}\,, \qquad R_{ij} = \frac{2\dot{a}}{a}\,\dot{S}\,\delta_{ij}\,, \qquad R_{\tilde{i}\tilde{j}} = \frac{2\dot{b}}{b}\,\dot{S}\,\delta_{\tilde{i}\tilde{j}}\,, \tag{5.39}$$

where the Ricci tensor is given by (5.34).

The first-order equations (5.38) can be solved by defining $u \equiv a\,b$, $v \equiv a/b$ and introducing a new radial variable r such that $dt = c\,dr$. The first-order equations become

$$v' + v^2 - 1 = 0\,, \qquad u' = c^2\,, \qquad \frac{c'}{c} + \frac{3u'}{2u} - \tfrac{3}{2}(v + v^{-1}) + S' = 0\,, \tag{5.40}$$

leading to the solution

$$v = \coth r\,, \qquad u^4 = \int_0^r e^{-2S(x)}\,(\sinh 2x)^3\,dx\,, \qquad c^2 = \tfrac{1}{4} e^{-2S}\,u^{-3}\,(\sinh 2r)^3\,. \tag{5.41}$$

In our perturbative discussion, we can solve explicitly for the linearised deformations by sending $S \longrightarrow \varepsilon\,S$, and writing

$$a = \bar{a}\,(1 + \varepsilon\,f)\,, \qquad b = \bar{b}\,(1 + \varepsilon\,f)\,, \qquad c = \bar{c}\,(1 + \varepsilon\,g)\,, \tag{5.42}$$

where the barred quantities denote the zero'th-order Ricci-flat expressions. These are given by [26]

$$\bar{a}^2 = R^{1/4}\,\coth r\,, \qquad \bar{b}^2 = R^{1/4}\,\tanh r\,, \qquad \bar{c}^2 = \tfrac{1}{4} R^{-3/4}\,(\sinh 2r)^3\,, \tag{5.43}$$

where

$$R \equiv \tfrac{3}{2}(2 + \cosh 2r)\,\sinh^4 r\,. \tag{5.44}$$

Solving for f and g at linearised order in ε, we then find

$$f = -\frac{\bar{P}}{\bar{a}^4\,\bar{b}^4}\,, \qquad g = \frac{3\bar{P}}{\bar{a}^4\,\bar{b}^4} - \bar{S}\,, \tag{5.45}$$

where

$$P \equiv \int_0^r a^3(x)\,b^3(x)\,c^2(x)\,S(x)\,dx\,, \tag{5.46}$$

and the quantities $\bar{P}$ and $\bar{S}$ are evaluated using the Riemann tensor in the undeformed Ricci-flat background.

For the α'^3 corrections, we find that f_3 and g_3 are given by

$$f_3 = \frac{80\,3^{\frac{3}{4}}\,e^{\frac{37}{2}r}}{(1+e^{2r})^{11}\,(1+4e^{2r}+e^{4r})^{\frac{15}{4}}}\Big(272488 + 434591\cosh 2r + 225766\cosh 4r$$

$$+78287\cosh 6r + 18120\cosh 8r + 2697\cosh 10r + 234\cosh 12r + 9\cosh 14r\Big)\,,$$

$$\begin{aligned}
g_3 \;&=\; \frac{160 e^{\frac{45}{2}r}}{3^{1/3}(1+e^{2r})^{15}\,(1+4e^{2r}+e^{4r})^{\frac{15}{4}}}\Big(9352650 + 13111232\cosh 2r + 3993140\cosh 4r \\
&\quad -415614\cosh 6r - 835680\cosh 8r - 343762\cosh 10r - 77940\cosh 12r \\
&\quad -10581\cosh 14r - 810\cosh 16r - 27\cosh 18r\Big)\,, \qquad (5.47)
\end{aligned}$$

In the comoving coordinate t, the metric functions a, b and c behave in the following way at small distances

$$\begin{aligned}
a &= \Big(2^{\frac{1}{8}} + \frac{280\, 2^{3/8}\,\epsilon}{3}\Big)\Big(1 + \tfrac{1}{6}(2^{3/4} - 2240\epsilon)\, t^2 + \cdots\Big)\,, \\
b &= t\,\Big(1 - (\frac{1}{6\, 2^{1/4}} - \frac{280\epsilon}{9})\, t^2 + \cdots\Big)\,, \\
c &= \Big(2^{\frac{1}{8}} + \frac{280\, 2^{3/8}\,\epsilon}{3}\Big)\Big(1 + (\frac{1}{2\, 2^{1/4}} - \frac{3080\epsilon}{3})\, t^2 + \cdots\Big)\,, \qquad (5.48)
\end{aligned}$$

and at large distances, they behaves as

$$\begin{aligned}
a &= \sqrt{\tfrac{3}{8}}\, t\,\Big(1 + \tfrac{4}{3}(\tfrac{2}{3})^{2/3}\, t^{-\frac{8}{3}} - \tfrac{80}{351}\,(\tfrac{2}{3})^{1/3}\, t^{-\frac{16}{3}} + 512\epsilon\, t^{-6} + \cdots\Big)\,, \\
b &= \sqrt{\tfrac{3}{8}}\, t\,\Big(1 - \tfrac{4}{3}(\tfrac{2}{3})^{2/3}\, t^{-\frac{8}{3}} - \tfrac{80}{351}\,(\tfrac{2}{3})^{1/3}\, t^{-\frac{16}{3}} + 512\epsilon\, t^{-6} + \cdots\Big)\,, \\
c &= \tfrac{3}{4} t\,\Big(1 + \tfrac{320}{117}(\tfrac{2}{3})^{1/3}\, t^{-16/3} - 2048\epsilon\, t^{-6} + \cdots\Big)\,. \qquad (5.49)
\end{aligned}$$

5.5 Corrections beyond α'^3 order

The calculation for higher-order corrections up to order α'^5 is straightforward, but the results are rather complicated to present in detail. We shall only list the large and small distance behaviour in the comoving coordinate system.

$U(1)$ bundle over $S^2 \times S^2 \times S^2$

As in the previous case, we only consider the simplest case with $a = b = c$. We just give the large and small distance behaviour. For $r \to 0$, we have

$$f = (\frac{1980\alpha'^3}{\ell^6} + \frac{10560\alpha'^4}{\ell^8} + \frac{233520\alpha'^5}{\ell^{10}})\,\frac{\rho}{\ell^2} - (\frac{21018\alpha'^3}{\ell^6} + \frac{153312\alpha'^4}{\ell^8} + \frac{4160920\alpha'^5}{\ell^{10}})\,\frac{\rho^2}{\ell^4} + \cdots\,. \qquad (5.50)$$

For $r \to \infty$, we have

$$f = \frac{9\alpha'^3}{\rho^3} + \Big(\frac{69\alpha'^3}{\ell^6} + \frac{3(1399 + 48\log(\rho/\ell^2))\alpha'^4}{16\ell^8} + \frac{987520\alpha'^5}{273\ell^{10}}\Big)\,\frac{\ell^8}{\rho^4} + \cdots\,. \qquad (5.51)$$

$U(1)$ bundle over $S^2 \times \mathbb{CP}^2$

In this case, the higher-order correction to the function s is given by (assuming the simple case $a=c$)

$$f=(\frac{1884\alpha'^3}{\ell^6}+\frac{95296\alpha'^4}{\ell^8}+\frac{6298160\alpha'^5}{\ell^{10}})\,\frac{\rho}{\ell^2}-(\frac{20886\alpha'^3}{\ell^6}+\frac{4147168\alpha'^4}{\ell^8}+\frac{37424680\alpha'^5}{\ell^{10}})\,\frac{\rho^2}{\ell^4}+\cdots\,. \tag{5.52}$$

For $r\to\infty$, we have

$$f=\frac{3\alpha'^3}{\rho^3}+\Big(\frac{87\alpha'^3}{\ell^6}+\frac{38173+6568\log(\rho/\ell^2))\alpha'^4}{144\ell^8}+\frac{8798080\alpha'^5}{2457\ell^{10}}\Big)\,\frac{\ell^8}{\rho^4}+\cdots\,. \tag{5.53}$$

<u>$U(1)$ bundle over $\mathbb{CP}^3$</u>

In this case, the correction is easy to obtain, since we have

$$S_3=\frac{495\alpha'^3\,X^3}{2a^6}\,,\qquad S_4=\frac{4245\alpha'^4\,X^4}{4a^8}\,,\qquad S_5=\frac{149145\alpha'^5\,X^5}{8a^{10}}\,, \tag{5.54}$$

where $X=(a^2-2g^2)/a^2$. The perturbation function f is given by

$$\begin{aligned} f &= 45\alpha'^3\Big(\frac{2}{r^3}+\frac{2\ell^8}{r^7}+\frac{2\ell^{16}}{r^{11}}+\frac{-11\ell^{24}}{2r^{15}}-\frac{1}{\ell^4\,(r+\ell^2)}+\frac{45(r-\ell^2)}{\ell^4\,(r^2+\ell^4)}\Big)\\ &\quad \tfrac{4245}{16}\alpha'^4\Big(\frac{1}{r^4}+\frac{\ell^8}{r^8}+\frac{\ell^{16}}{r^{12}}+\frac{\ell^{24}}{r^{16}}-\frac{4\ell^{32}}{r^{20}}\Big)+\tfrac{49715}{56}\alpha'^5\Big(\frac{4}{\ell^8\,r}+\frac{4}{r^5}\\ &\quad +\frac{4\ell^8}{r^9}+\frac{4\ell^{16}}{r^{13}}+\frac{4\ell^{24}}{r^{17}}+\frac{4\ell^{32}}{r^{21}}-\frac{21\ell^{40}}{r^{25}}-\frac{2}{\ell^8\,(r+\ell^2)}-\frac{2(r+\ell^2)}{\ell^8\,(r^2+\ell^4)}\Big)\,, \end{aligned}$$

where $r=\rho-\ell^2$.

<u>Stenzel metric</u>

The structure in this case is again rather complex. We shall only present S_3, S_4 and S_5, which are given by

$$\begin{aligned} S_3 &= -\frac{80e^{\frac{45}{2}r}}{3^{1/4}(1+e^{2r})^{15}\,(1+4e^{2r}+e^{4r})^{15/4}}\Big(51096822+82052992\cosh 2r\\ &\quad +43709116\cosh 4r+16111758\cosh 6r+4256544\cosh 8r+823522\cosh 10r\\ &\quad +117252\cosh 12r+12021\cosh 14r+810\cosh 16r+27\cosh 18r\Big)\\ S_4 &= -\frac{1}{37748736\cosh^{20}r\,(2+\cosh 2r)^5}\Big(1912969150222+3256894409584\cosh 2r\\ &\quad +2031561604552\cosh 4r+949210599696\cosh 6r+339746197983\cosh 8r\\ &\quad +94849519304\cosh 10r+20896666964\cosh 12r+3648186232\cosh 14r\\ &\quad +501788290\cosh 16r+53212824\cosh 18r+4135140\cosh 20r+210600\cosh 22r\\ &\quad +5265\cosh 24r\Big) \end{aligned}$$

$$
\begin{aligned}
S_5 \;=\; & -\frac{160 e^{\frac{75}{2} r}}{9\, 3^{3/4}\, (1+e^{2r})^{25}\, (1+4e^{2r}+e^{4r})^{25/4}} \Big(3192260095227860 \\
& +5621631779278675 \cosh 2r + 3857631050654430 \cosh 4r \\
& +2088920390620110 \cosh 6r + 906177970153450 \cosh 8r \\
& +319423127230328 \cosh 10r + 92531981651010 \cosh 12r + 22191184070115 \cosh 14r \\
& +4418078655500 \cosh 16r + 728378998045 \cosh 18r + 98534694870 \cosh 20r \\
& +10738276020 \cosh 22r + 912024630 \cosh 24r + 56882790 \cosh 26r \\
& +2320650 \cosh 28r + 46413 \cosh 30r\Big)
\end{aligned}
\tag{5.55}
$$

From these, it is straightforward to find the perturbation functions f and g, given in (5.45).

6 Conclusion

In this paper, we have shown how the preservation of supersymmetry on BPS backgrounds such as non-compact Calabi-Yau spaces can be used to obtain explicit expressions for the string-theory-derived α' corrections to these backgrounds. The corrected Killing spinor conditions are the key to this. Even in the absence of full knowledge of the supersymmetric structure of the α' corrections, these Killing spinor conditions can be deduced from the requirement that the corrected bosonic effective field equations appear as integrability conditions for them. It is to be hoped that these corrected conditions may illuminate the problem of supersymmetrising the string-theory corrections, and in particular the important quartic curvature corrections arising at order ${\alpha'}^3$, for which partial results have been given in [14, 15, 16].

For Kähler manifolds, the scheme adopted in this paper has the virtue of preserving the Kähler structure, although the Ricci-flatness of the space is necessarily lost, since the deformed space develops a new $U(1)$ factor in its holonomy. At the same time, the dilaton ϕ acquires corrections as given in Eqn (2.28). This latter point is of little significance, since it can clearly be reset at order ${\alpha'}^3$ by defining a new dilaton $\tilde{\phi}$ that is related to ϕ by

$$
\tilde{\phi} = \phi + \tfrac{1}{2}{\alpha'}^3\, S_3\,. \tag{6.1}
$$

A consequence of this redefinition is merely to change the specific form of the ${\alpha'}^3$ corrections in the effective Lagrangian. One of these changes is a modification of the coefficient of $Y_2^{(2)}$ in (2.31). Moreover, as noted previously, this coefficient can be adjusted by field redefinitions in a sigma-model calculation of the effective action. If one wants to avoid altering this

coefficient, one can achieve this by making a compensating transformation of the metric; for example by sending $g_{ab} \longrightarrow \tilde{g}_{ab}$ with

$$\tilde{g}_{ab} = e^{-\frac{1}{8}\alpha'^3 S_3}\, g_{ab} \tag{6.2}$$

at the same time as $\phi \longrightarrow \tilde{\phi}$ The change in the Ricci tensor under a Weyl transformation $g_{ab} \rightarrow \tilde{g}_{ab} = e^{2\sigma} g_{ab}$ in a space of dimension D is

$$\tilde{R}_{ab} = R_{ab} - (D-2)\nabla_a \nabla_b \sigma - (D-1)(D-2)\nabla_a \sigma \nabla_b \sigma - \Box \sigma g_{ab}\ , \tag{6.3}$$

so, setting $D = 10$ and keeping terms only to order α'^3, the corrected Einstein equation (3.1) becomes

$$\widetilde{R}_{ij} = \alpha'^3 (\widetilde{\nabla}_{\hat{i}} \widetilde{\nabla}_{\hat{j}} \widetilde{S}_3 - \tfrac{1}{8} \widetilde{\Box} \widetilde{S}_3\, \tilde{g}_{ij}) \tag{6.4}$$

in the conformally-related metric. The price to be paid for doing this is that the metric $\tilde{g}_{ij}$ is no longer Kähler.

A similar type of field redefinition, but expressible in terms of a purely six-dimensional Weyl scaling, takes one from the Kähler-preserving scheme employed in this paper to the scheme used in [2]. From a geometrical point of view a scheme that preserves the Kähler structure of the metric is appealing. Schemes that do not preserve the Kähler structure would appear to have a more *ad hoc* character.

The technique for obtaining explicit expressions for α' corrections to internal manifolds employed in this paper extends naturally to $D = 7$ manifolds with G_2 holonomy. This is discussed separately in Ref. [33].

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APPENDICES

A Topological Invariants and the *Curvatura Integra*

The Euler number of a compact manifold M of (even) dimension $n = 2p$ is given by integrating the n form

$$\Psi \equiv \frac{1}{p!\,(4\pi)^p}\,\epsilon^{a_1b_1\cdots a_pb_b}\,\Theta_{a_1b_1}\wedge\cdots\wedge\Theta_{a_pb_p}\,, \tag{A.1}$$

where $\Theta_{ab} = d\omega_{ab} + \omega_a{}^c\wedge\omega_{cb}$ is the curvature 2-form; $\chi = \int_M \Psi$. The n-form Ψ can be rewritten as $\Psi = E_n\,\sqrt{g}\,d^nx$, where the "Euler integrand" E_n is given by

$$E_n = \frac{(2p-1)!!}{(4\pi)^p}\,R_{a_1a_2}{}^{[a_1a_2}\,R_{a_3a_4}{}^{a_3a_4}\cdots R_{a_{n-1}a_n}{}^{a_{n-1}a_n]}\,. \tag{A.2}$$

In a non-compact manifold, the Euler number is not given just by the volume integral of the Euler integrand; there is also a boundary term that must be included [32]:

$$\chi = \int_M \Psi + \int_{\partial M}\Phi\,, \tag{A.3}$$

where in $n = 2p$ dimensions the *Curvatura Integra* Φ is an $(n-1)$-form constructed from the Riemann curvature and the second fundamental form of the boundary. It is shown in [32] that if u^a denotes the unit outward-pointing vector normal to the boundary, then Φ is given by

$$\Phi = \frac{1}{(2\pi)^p}\sum_{m=0}^{p-1}\frac{2^{-m}}{m!\,(2p-2m-1)!!}\,\Phi_{(m)}\,, \tag{A.4}$$

where

$$\Phi_{(m)} = \epsilon^{ab_1\cdots b_{n-2m-1}c_1d_1\cdots c_md_m}\,u_a\,\theta_{b_1}\wedge\cdots\wedge\theta_{b_{n-2m-1}}\wedge\Theta_{c_1d_1}\wedge\cdots\wedge\Theta_{c_md_m}\,. \tag{A.5}$$

The second fundamental form is defined by

$$\theta_a = Du_a \equiv du_a + \omega_{ab}\,u^b\,. \tag{A.6}$$

In the case of metrics $ds^2 = dt^2 + d\bar{s}^2(t)$, which includes all our examples in sections 4 and 5, the unit vector normal to the boundary at $t = t_0$ is just given by $u = \partial/\partial t$, and so we shall have $u_0 = 1$, $u_i = 0$ for $i \ge 1$. Thus we have

$$\theta_0 = 0\,,\qquad \theta_i = -\omega_{0i}\,,\quad i \ge 1\,. \tag{A.7}$$

In six dimensions, equation (A.4) gives

$$\Phi = \frac{1}{8\pi^3}\Big[\tfrac{1}{15}\,\Phi_{(0)} + \tfrac{1}{6}\Phi_{(1)} + \tfrac{1}{8}\Phi_{(2)}\Big]\,, \tag{A.8}$$

and (A.5) gives

$$
\begin{aligned}
\Phi_{(0)} &= \epsilon^{ijk\ell m}\,\theta_i\wedge\theta_j\wedge\theta_k\wedge\theta_\ell\wedge\theta_m\,,\\
\Phi_{(1)} &= \epsilon^{ijk\ell m}\,\theta_i\wedge\theta_j\wedge\theta_k\wedge\Theta_{\ell m}\,,\\
\Phi_{(2)} &= \epsilon^{ijk\ell m}\,\theta_i\wedge\Theta_{jk}\wedge\Theta_{\ell m}\,.
\end{aligned}
\tag{A.9}
$$

In eight dimensions, the corresponding expressions are given by

$$
\Phi=\frac{1}{16\pi^4}\Big[\tfrac{1}{105}\,\Phi_{(0)}+\tfrac{1}{30}\Phi_{(1)}+\tfrac{1}{24}\Phi_{(2)}+\tfrac{1}{48}\Phi_{(3)}\Big]\,,\tag{A.10}
$$

with

$$
\begin{aligned}
\Phi_{(0)} &= \epsilon^{ijk\ell mpq}\,\theta_i\wedge\theta_j\wedge\theta_k\wedge\theta_\ell\wedge\theta_m\wedge\theta_p\wedge\theta_q\,,\\
\Phi_{(1)} &= \epsilon^{ijk\ell mpq}\,\theta_i\wedge\theta_j\wedge\theta_k\wedge\theta_\ell\wedge\theta_m\wedge\Theta_{pq}\,,\\
\Phi_{(2)} &= \epsilon^{ijk\ell mpq}\,\theta_i\wedge\theta_j\wedge\theta_k\wedge\Theta_{\ell m}\wedge\Theta_{pq}\,,\\
\Phi_{(3)} &= \epsilon^{ijk\ell mpq}\,\theta_i\wedge\Theta_{jk}\wedge\Theta_{\ell m}\wedge\Theta_{pq}\,.
\end{aligned}
\tag{A.11}
$$

It is interesting to note that if we vary the metric g_{ab} in E_8 then those terms linear in R_{ab} are given by

$$
\delta E_8=\frac{1}{4\pi}\,E_6\,R_{ab}\,\delta g^{ab}\,,\tag{A.12}
$$

where E_6 is precisely the Euler integrand of six dimensions (including all Ricci-tensor terms),

$$
E_6=\frac{15}{(4\pi)^3}\,R_{a_1a_2}{}^{[a_1a_2}\,R_{a_3a_4}{}^{a_3a_4}\,R_{a_5a_6}{}^{a_5a_6]}\,.\tag{A.13}
$$

The cubic curvature invariant S_3 given in (2.10) is proportional, modulo terms involving the Ricci tensor, to the Euler integrand E_6 in six dimensions. The exact expression for E_6, including all Ricci terms, is

$$
\begin{aligned}
E_6 &\equiv \frac{15}{64\pi^3}\,R_{a_1a_2}{}^{[a_1a_2}\,R_{a_3a_4}{}^{a_3a_4}\,R_{a_5a_6}{}^{a_5a_6]}\,,\\
&= \frac{S_3}{96\pi^3}+\frac{1}{384\pi^3}\,(-24R_{abcd}\,R^{abc}{}_e\,R^{de}+3R\,R_{abcd}\,R^{abcd}+24R^{abcd}\,R_{ac}\,R_{bd}\\
&\qquad +16R_a{}^b\,R_b{}^c\,R_c{}^a-12R\,R_{ab}\,R^{ab}+R^3)\,.
\end{aligned}
\tag{A.14}
$$

Thus we see that when evaluated in the Ricci-flat unperturbed Calabi-Yau metric, we shall have

$$
\frac{1}{96\pi^3}\int_M S_3\,\sqrt{g}\,d^6x\equiv\widetilde{\chi}=(\chi-\Xi)\,,\qquad\text{where}\quad \Xi\equiv\int_{\partial M}\Phi\,.\tag{A.15}
$$

Here Ξ is the contribution to the Euler number from the surface term in (A.3). Thus the quantity $\widetilde{\chi}$ that results from integrating S_3 over a non-compact Calabi-Yau manifold is neither the Euler number nor is it a topological invariant.

B Euler Numbers from Curvature Integrals

In Appendix A, we review some standard material on the calculation of the Euler number in terms of integrals over quantities formed from the curvature of the metrics. Because the manifolds M that we are studying here are non-compact, it is necessary to include the contributions not only of the usual volume term in the Euler integrand, but also a contribution coming from the boundary ∂M that one can introduce in order to compactify the manifold. The Euler number is then given by

$$\chi = \int_{M_0} \Psi + \int_{\partial M_0} \Phi \,, \tag{B.1}$$

where M_0 denotes the compact manifold introduced by cutting off the n-dimensional non-compact manifold M with a boundary ∂M_0. The answer is, of course, independent of any smooth deformation of ∂M_0. It is useful to introduce the notation ∂M to denote the limiting case where the boundary is pushed out all the way to infinity. The n-form Ψ is the usual Euler form, and the $(n-1)$-form Φ is the *curvatura integra* that is constructed in [32], which supplies the boundary term.

It is now a mechanical exercise to calculate the contributions given in (B.1) to the Euler number for each of the manifolds we have considered here. Considering first the six-dimensional cases in section 4, we find

$$\begin{aligned}
&\textbf{Resolved Conifold}: && \int_M \Psi = \tfrac{14}{27}\,, && \int_{\partial M} \Phi = \tfrac{40}{27}\,, && \chi = \tfrac{14}{27} + \tfrac{40}{27} = 2\,,\\
&\textbf{Deformed Conifold}: && \int_M \Psi = -\tfrac{40}{27}\,, && \int_{\partial M} \Phi = \tfrac{40}{27}\,, && \chi = -\tfrac{40}{27} + \tfrac{40}{27} = 0\,,\\
&\mathbb{R}^2 \textbf{ bundle over } S^2 \times S^2: && \int_M \Psi = \tfrac{88}{27}\,, && \int_{\partial M} \Phi = \tfrac{20}{27}\,, && \chi = \tfrac{88}{27} + \tfrac{20}{27} = 4\,,\\
&\mathbb{R}^2 \textbf{ bundle over } \mathbb{CP}^2: && \int_M \Psi = \tfrac{8}{3}\,, && \int_{\partial M} \Phi = \tfrac{1}{3}\,, && \chi = \tfrac{8}{3} + \tfrac{1}{3} = 3\,,
\end{aligned} \tag{B.2}$$

where the boundary is taken to be at $t = t_0$, in the limit where $t_0 \longrightarrow \infty$. These results are all consistent with expectation. The resolved conifold is an $\mathbb{R}^4$ bundle over S^2, whose Euler number is the same as that for a direct product $\mathbb{R}^4 \times S^2$, giving $\chi = 1 \times 2 = 2$. The deformed conifold is an $\mathbb{R}^3$ bundle over S^3, giving $\chi = 1 \times 0 = 0$. The $\mathbb{R}^2$ bundles over $S^2 \times S^2$ and $\mathbb{CP}^2$ give $\chi = 1 \times 2 \times 2 = 4$ and $\chi = 1 \times 3 = 3$ respectively.

It should be noted that even in a case such as the deformed conifold, which has zero Euler number, the volume integral of the Euler integrand E_6 is non-zero.

We now turn to the eight-dimensional metrics that we considered in section 5. For these,

we find

$$
\begin{aligned}
\mathbb{R}^2 \textbf{ bundle over } S^2\times S^2\times S^2: &\quad \int_M \Psi = \tfrac{111}{16}\,, \qquad \int_{\partial M}\Phi = \tfrac{17}{16}\,, \qquad \chi = 8\,,\\
\mathbb{R}^4 \textbf{ bundle over } S^2\times S^2: &\quad \int_M \Psi = \tfrac{15}{8}\,, \qquad \int_{\partial M}\Phi = \tfrac{17}{8}\,, \qquad \chi = 4\,,\\
\mathbb{R}^2 \textbf{ bundle over } S^2\times \mathbb{CP}^2: &\quad \int_M \Psi = \tfrac{687}{128}\,, \qquad \int_{\partial M}\Phi = \tfrac{81}{128}\,, \qquad \chi = 6\,,\\
\mathbb{R}^4 \textbf{ bundle over } \mathbb{CP}^2: &\quad \int_M \Psi = \tfrac{111}{64}\,, \qquad \int_{\partial M}\Phi = \tfrac{81}{64}\,, \qquad \chi = 3\,,\\
\mathbb{R}^2 \textbf{ bundle over } \mathbb{CP}^3: &\quad \int_M \Psi = \tfrac{15}{4}\,, \qquad \int_{\partial M}\Phi = \tfrac{1}{4}\,, \qquad \chi = 4\,, \qquad \text{(B.3)}
\end{aligned}
$$

Again, these Euler numbers accord with one's expectations, since the Euler number for a fibre $\mathbb{R}^m$ over a base B is just given by the Euler number of B, and we know that $\chi(S^2)=2$, $\chi(\mathbb{CP}^2)=3$ and $\chi(\mathbb{CP}^3)=4$.

Note that although one customarily tends to evaluate the volume and boundary contributions to the Euler number by choosing a boundary surface that is pushed out to infinity, as in our results presented above, the boundary can equally well be chosen to be at any radius. We have explicitly verified for all the six-dimensional and eight-dimensional examples listed above that one indeed gets the identical results for $\int_M \Psi + \int_{\partial M}\Phi$ when the bounding surface is taken to be at any radius r_0. This provides a useful check that the computations of Ψ and Φ, which are quite involved, are indeed correct.

An interesting limiting choice for the radius of the bounding surface is to take it to lie at $r_0 = 0$; i.e. at the origin, on the base B of the $\mathbb{R}^n$ fibre bundle over B. In this case, there is no contribution at all from the volume integral $\int_M \Psi$, and the entire contribution to the Euler number comes from the boundary term $\int_{\partial M}\Phi$, with Φ evaluated at $r=0$.

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